religion (Hermetism), which he viewed as the revival of the pristine and true religion" (French, Ref. 36, p. 103). Bruno would have been motivated towards heliocentricity for the same reasons and, being bolder, he would not have hesitated to pronounce his new allegiance.

[47] *Op. It.* I, p. 126.

[48] *Op. It.* I, pp. 118–122. We have reason to suspect that Bruno's geological uniformitarianism is related to his concept of the doctrines of plenitude and sufficient reason as they are manifested in the chain of being. On the role of these two doctrines in Bruno's thought, see A. Lovejoy, *The Great Chain of Being* (Harvard U. P., Cambridge, 1971), pp. 116–121. We believe that Bruno was anticipatory of the 18th century "temporalizing" of the chain of being, as described in Lovejoy, *ibid.*, chap. 9 *passim;* we hope to develop this idea more fully elsewhere.

[49] *Op. It.* I, pp. 97–99.

[50] "Well then, if the earth moves, does Mars appear now bigger, now smaller?" Torquato expresses himself mostly in Latin.

[51] Bruno does not notice that this remark flies in the face of his prior argument for the constancy of the brightness of Venus. Alternatively, one may argue that he does realize this fact, and introduces the remark purposely in preparation for his subsequent expressions of his disinterest in the details of astronomical models.

[52] Yates, Ref. 4, p. 241.

[53] P. O. Kristeller, *Eight Philosophers of the Italian Renaissance* (Stanford U.P., Stanford, 1964), p. 138.

Einstein's Derivation of Planck's Radiation Law

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Einstein's derivation, almost lost through distorted referencing, still contains insights and problems worth reconsidering after more than 40 years.

INTRODUCTION

In 1925, Eddington[1] began an article in the *Philosophical Magazine* with the following observation: "It is generally considered that Einstein's[2] investigation in Physikalische Zeitschrift (1917) gives the clearest insight into the origin of Planck's law of radiation." Because of this general appreciation of the substance of Einstein's argument, an interesting literature grew around it. Now, more than 50 years after Einstein's article was written, it is still frequently cited, but in almost all cases the context of the reference is not to that "insight into the origin of Planck's law" but instead to a discussion of the coefficients of stimulated and spontaneous emission which bear Einstein's name. Planck's formula is assumed in these recent summaries, and Einstein's detailed balance argument is used only to evaluate the ratio of spontaneous to stimulated emission. It is probably true that few physicists now know that the often cited aritcle is, in fact, a derivation of Planck's law.[3]

If only for historical reasons, it seems appropriate to restate once again Einstein's derivation. Beyond that, the argument is made here that the "insight" mentioned by Eddington is greater than has been appreciated. Einstein's derivation includes in addition to purely classical assumptions, a reference to a quantum theory hypothesis and to discrete states. Yet his derivation does not require discrete states, and the Bohr frequency relation is a result of this derivation, not an assumption. When such a simple derivation gives quantum results as fundamental as the blackbody radiation law and the Bohr frequency condition, it is important to state clearly the simplest nonclassical assumption which is required.

In addition, Appendices A–C summarize the results of three related detail balance arguments, two of them from the 1920's. They illustrate the power of such proofs and highlight the generality of Einstein's rule for the ratio of A and B coefficients in a way that can simplify for the experimental physicist some otherwise very complicated calculations.

EINSTEIN'S DERIVATION OF BLACKBODY RADIATION

Einstein considered the exchange of energy between thermal radiation and an ensemble of fixed atoms. He assumed: (1) the existence of a discrete set of energy states for the atoms, (2) a Boltzmann distribution for the atoms among those states, (3) Wien's displacement law, $\rho(\nu, T)=a\nu^3 f(\nu/T)$, for the energy density of thermal radiation, and (4) particular forms for the absorption and emission of radiant energy by atoms. (The reader should note the use of Wien's displacement *law* that is deduced from classical thermodynamics,[4] and not Wien's radiation *hypothesis* with which it is sometimes confused.) For the probability per unit time that an atom with energy ϵ_i absorbs energy and changes to a state with energy ϵ_j, Einstein assumed the form

$$dW=B_i{}^j\rho dt, \tag{1}$$

where $\rho(\nu, T)$ is the radiation density. For the inverse process in which radiant energy of the same frequency and polarization is emitted by the atom into the field while the atom changes from state j to state i, Einstein assumed the form

$$dW=B_j{}^i\rho dt+A_j{}^i dt. \tag{2}$$

The constants A and B can be functions of the atomic configuration and the radiation frequency, but not of temperature. In Eqs. (1) and (2) the terms involving ρ are the radiation induced transitions, and the remaining term describes emission which is not externally excited. In the past 10 years since the demonstration of the maser, many physicists brought up on quantum theory have had to grope for the physical meaning of stimulated emission. Forty years earlier physicists who were trying to relate classical mechanics and electrodynamics to atomic phenomena apparently regarded the stimulated term as a reasonable consequence of any mechanical model of the atom. Einstein's contribution was not to formulate stimulated emission, but to put the argument into a probabilistic form and to show the force of detailed balance arguments.

The rest of the derivation is very simple. Detailed balance between the allowed states requires that the net rate at which atoms undergo the process described by Eq. (1) be equal to the net rate of the inverse process, described by Eq. (2). The net rate for each process is found by multiplying the rate per atom by the number of atoms in the initial state at thermal equilibrium. If $p_i \exp(-\epsilon_i/kT)$ is the relative number of atoms with energy ϵ_i according to the Boltzmann distribution, then a detailed balance between the processes of Eqs. (1) and (2) requires

$$p_i \exp(-\epsilon_i/kT) B_i{}^j\rho = p_j \exp(-\epsilon_j/kT)(B_j{}^i\rho + A_j{}^i).$$

If the radiation density increases without bound as temperature increases,

$$p_i B_i{}^j = p_j B_j{}^i. \tag{3}$$

Then detailed balance requires

$$\rho=\frac{A_j{}^i/B_j{}^i}{\exp[(\epsilon_j-\epsilon_i)/kT]-1}. \tag{4}$$

Since A and B are independent of temperature, Wien's displacement law shows that

$$A_j{}^i/B_j{}^i=\alpha\nu^3,$$
$$\epsilon_j-\epsilon_i=h\nu, \tag{5}$$

where the numerical values of the constants α and h are yet to be determined. This, apart from the numerical values of the constants α and h, completes the derivation of Planck's law for blackbody radiation. Einstein, of course, immediately identified the relation between energy and frequency with Bohr's hypothesis which Einstein had previously used to explain the photoelectric effect. Here, however, the relation is a result of this derivation. We know from Kirchoff's law that both α and h must be independent of the material in equilibrium with the radiation. Einstein also pointed out that although we cannot derive an expression for α without a more detailed knowledge of the interaction of light and matter, $\alpha = 8\pi h/c^3$ can be obtained by identification of Eq. (4) at high temperatures with the semiempirical Jeans formula

$$\rho = 8\pi kT\nu^2/c^3.$$

The arguments outlined above are the most elegant and important part of the 1917 Einstein paper. The paper also shows that during each elementary process, a momentum $h\nu/c$ must be exchanged if the atoms are to remain in thermal equilibrium. In the case of stimulated processes, this momentum is in the direction of propagation of the stimulating radiation. The argument was intended to provide further foundation for the existence, in some particlelike sense, of bundles of energy $h\nu$ in an electromagnetic field.

DISCUSSION

An interesting question that arises from the proof is exactly where the quantum character is inserted into this simple derivation of the world's first quantum equation. Superficially it might appear to be in the assumption of discrete energy states for the atom. However, by 1923 Pauli[5] and Einstein[6] were to point out that the logic of the proof applies equally well to radiation in equilibrium with free electrons, i.e., particles with continuous allowed energy states. The key point is not the assumption of discrete states, but rather that the atom is almost always in some particular energy state and spends negligible time in changing its energy. It is that view which underlies Eqs. (1)–(3) and which introduces the nonclassical element.

In place of this nonclassical assumption about the atoms, we can make a nonclassical assumption about the electromagnetic field only. A sufficient assumption is to take Newton's view that a monochromatic wave is made up of discrete particles. Then absorption and emission are naturally thought of as the result of collisions between light particles and atoms. The atoms are isolated except for negligibly short times during which an interaction takes place. In that way, Eqs. (1)–(3) can be justified. This is analogous to deriving the Boltzmann distribution by using detailed balance and conservation of energy for an atomic gas (Appendix A). In that case also one assumes negligible interaction time.

We now retain our wave description of thermal radiation and the proof follows identically as before. Simply by considering a grainy substructure (photons) to the electromagnetic wave, the detailed balance argument forces the following conclusions: (1) Radiation of frequency ν can produce a change from one energy state to another only if the change in energy is proportional to ν. The proportionality constant is a universal constant. (2) A/B is proportional to ν^3, and the proportionality constant is a universal constant. (3) Conclusion (1) implies that each photon carries energy $h\nu$. Then the number of photons per classically allowed mode is

$$[\exp(h\nu/kT) - 1]^{-1} = [\exp(\epsilon/kT) - 1]^{-1}.$$

That is, Bose–Einstein statistics applies to photons.

CONCLUSION

The nonclassical assumption in Einstein's derivation of Planck's black-body radiation formula is that of stationary energy states. An alternative, nonclassical assumption is the concept of light waves made up of a substructure of particles (photons). It is interesting that such an assumption about the nature of light is sufficient to impose quantized energy levels on atoms in equilibrium with light. In addition, one uses the Wien displacement law and an assumption that spontaneous emission is described by Eq. (2).

The photon assumption as a basis for the derivation is interesting because then the Planck formula is derived by adding something to the classical picture rather than by postulates in conflict with classical concepts. It is not surprising that classical theory, with its inability to cope with the problem of the structure of charged particles, does not give correct answers to some problems of the interaction of light with particles. If one assumes photons, it allows one to get some answers without considering the detailed interaction. This procedure is similar to modern scattering theories. The view that the Rayleigh–Jeans formula is a strict consequence of classical theory, and therefore Planck's formula must contradict classical theory, is not correct. That depends on identifying the infinite spectrum of standing waves in space with an infinite number of degrees of freedom of a material body. That is not at all an obvious conclusion (Jeans only suggested it as an interesting way to proceed), and it is more accurate to describe classical theory as too incomplete to deal with a derivation of thermal radiation.

An important aspect of the work of Einstein, Dirac, Pauli, and others, which is not well known, at least to experimentalists, is the following general law: *Any* process that results in emission of radiation is also stimulated by external radiation of the same frequency. The ratio of stimulated to spontaneous radiation is a universal constant times the intensity of the incident radiation divided by its frequency cubed, and the direction of the stimulated emission is the same as that of the spontaneous emission. This rule applies, for example, to the scattering of light by free electrons, including both Rayleigh and Compton scattering. Appendix B summarizes Einstein's paper on this subject while Appendix C contains a summary of Eddington's interesting extension of Einstein's 1917 paper.

APPENDIX A

The detailed balance argument sometimes used to derive Boltzmann statistics provides an interesting, purely classical analog of Einstein's argument. Let $f(\epsilon, P_i, T)$ be the distribution function for free atoms in thermal equilibrium. We consider a collision of two atoms of classes 1 and 2, resulting in atoms of classes 3 and 4 and describe the rate at which the process proceeds:

$$f(\epsilon_1, p_{i1}, T)f(\epsilon_2, p_{i2}, T)\,VB_{1,2}{}^{3,4}, \qquad (A.1)$$

where V is their relative velocity. If the collision is elastic, the inverse process will be

$$f(\epsilon_3, p_{i3}, T)f(\epsilon_4, p_{i4}, T)\,VB_{3,4}{}^{1,2} \qquad (A.2)$$

If we argue that detailed balance is required for thermal equilibrium, we can equate (A.1) and (A.2). If we consider the effect of time reversal on isotropic atoms in isotropic space, we can show that $B_{1,2}{}^{3,4}=B_{3,4}{}^{1,2}$. Then

$$f(\epsilon_1, p_1, T)f(\epsilon_2, p_2, T)=f(\epsilon_3, p_3, T)f(\epsilon_4, p_4, T). \qquad (A.3)$$

Equation (A.3) is usually derived by proving Boltzmann's H theorem, which is equivalent to proving detailed balance for elastic collisions. Equation (A.3) along with conservation of energy and momentum allows only solutions of the form

$$f(\epsilon, p, T)=A\exp(B\epsilon+\sum_i C_i p_i). \qquad (A.4)$$

The arbitrary constants are determined by knowledge of the total energy and momentum of the system together with some definition of temperature. If the net momenta of the ensemble are zero, the C_i's are zero. To complete the analogy with Einstein's derivation, one can use a theorem for atoms which is the counterpart of Wien's displacement law and may be derived by the same kind of consideration—atoms in an enclosure with an elastically reflecting piston.

The derivation just given is purely classical. How does it differ from Einstein's? A key, (but implicit) assumption in this derivation of the Boltzmann law is a negligible interaction time for the collision. This assumption is the counterpart of the stationary state assumption for the Einstein proof. We have argued that an alternative and simpler nonclassical assumption is that of the particle nature of light together with the assumption that Eqs. (1) and (2) have the correct form for absorption and emission of light.

APPENDIX B

Equilibrium Between Radiation and Free Electrons

Although Planck's law gave a far better fit to measurements than any previous formula, there were still experimental questions about the precision of the fit as late as 1920. This situation was, of course, complicated by the great problems the derivation of the Planck law posed to conventional theory. Even accepting the need for a new mechanics for atoms, and thus some surprises in their interaction with an electromagnetic field, there remained the question of the interaction of thermal radiation with "classical" systems. Lorentz (in 1911) and Focker (in 1912) considered the equilibrium of free electrons with thermal radiation. They showed that according to classical electrodynamics, if one postulates an external mechanism for maintaining radiation in a Planck distribution, the electrons will have a mean kinetic energy different from $3/2kT$, the value required by classical theory. Since Einstein's original article emphasized the discrete energy levels of atoms (unnecessarily as it turned out) this appeared to be a new problem. Pauli,[5] in 1923, undertook an analysis of the problem to see what modifications must be made in a theory of the interaction of an electromagnetic field and light so that the correct thermal distributions of radiation and free electrons are maintained. He called this a "quantum" modification. Pauli examined the requirements of detailed balance under Lorentz transformations and found that scattering of light by free electrons must include a term of a form which we would now call stimulated emission. This is true even when the scattered light is at a frequency different from the incident light, leading to an interaction in which the probability per electron of a scattering event takes the surprising form,

$$dW = (A\rho + B\rho\rho')dt, \qquad \text{(B.1)}$$

where ρ is the radiation density at the frequency of the incident radiation, and ρ' is the density at the frequency of the scattered radiation. Einstein and Ehrenfest[6] then showed that Pauli's results could be obtained by an extension of the 1917 paper with the unnecessary specialization to discrete energy levels removed. The Einstein argument is simpler and more in line with the discussions of this paper and will therefore be used here. However, the Pauli article is perhaps clearer in outlining the problem.

The core of Einstein's argument is that the scattering process should be broken into two parts: the absorption of energy from radiation of frequency ν_1 and the emission of energy as radiation of frequency ν_2. He then can apply his rate Eqs. (1) and (2) to these processes. (Since the intermediate state here is virtual, this procedure is perhaps not so obvious as his article suggests.) Generalizing further, Einstein wrote the general rate equation for any process in which radiation at frequencies $\nu_1, \nu_2, \cdots, \nu_n$ is absorbed and radiation at frequencies $\nu_1', \nu_2', \cdots, \nu_n'$ is emitted. This generalization of (1) and (2) is

$$dW = \prod b_i\rho_i \prod (a_i' + b_i'\rho_i')dt, \qquad \text{(B.2)}$$

where each pair of coefficients a_i, b_i depends on the initial and final states of the electron involved in the partial event, the frequency of the radiation emitted or absorbed in that partial event, and the direction of the radiation. In this formulation it is assumed at the outset from the analysis of the special case treated by the 1917 paper that the coefficients of absorption and stimulated emission are equal. The probability of the inverse process is

$$dW = \prod (a_i + b_i\rho_i) \prod b_i'\rho_i'dt. \qquad \text{(B.3)}$$

Then, if at equilibrium there are n and n^* electrons in the initial and final states of Eq. (B.2), detailed balance requires

$$n \prod b_i\rho_i \prod (b_i'\rho_i' + a_i') = n^* \prod b_i'\rho_i' \prod (b_i\rho_i + a_i). \qquad \text{(B.4)}$$

Since Boltzmann statistics require that

$$n/n^* = \exp[h(\sum \nu_i - \sum \nu_i')kT],$$

Eq. (B.4) may be written as

$$\frac{\prod [b_i\rho_i/(a_i + b_i\rho_i)] \exp(h\nu_i/kT)}{\prod [b_i'\rho_i'/(a_i' + b_i'\rho_i')] \exp(h\nu_i'/kT)} = 1. \qquad \text{(B.5)}$$

Einstein and Ehrenfast then observe that if $a_i/b_i = 8\pi h\nu^3/c^3$ and ρ is the Planck distribution, every term in both of the products of Eq. (B.5) is identically unity and Eq. (1) is satisfied. Using that condition and Wien's law, one can solve for ρ_i and derive that Planck distribution as before. If Eq. (B.2) is specialized to absorption of energy at frequency ν and emission at frequency ν', as in the case considered by Pauli,

$$dW=(a'b\rho+bb'\rho\rho')\,dt,$$

and we have identified the coefficients appearing in (B.1) with products of the coefficients for the elementary processes. It is now clear also that in Eq. (B.1)

$$A/B=8\pi h\nu^3/c^3,$$

i.e., that the stimulated emission occurring in the process of scattering by free electrons (or in any other "partial" process emitting electromagnetic energy) bears the same relation to scattering with no external electromagnetic field that stimulated radiation bears to spontaneous radiation from an allowed excited state of a system with internal degrees of freedom (e.g., an atom). This relation was used by Kapitza and Dirac[7] in 1933 in a discussion of electron scattering by a standing wave. It can, of course, also be obtained from a conventional quantum mechanical treatment.[8] Pauli's general rule for stimulated emission in all processes emitting electromagnetic energy does not appear to be as well known as it should be.

APPENDIX C

Eddington's Modification of Einstein's Derivation

In 1925, Eddington[1] showed how Planck's law could be obtained without assuming Boltzmann's formula by an extension of Einstein's argument. Eddington assumed both Wien's law, and Bohr's law, $h\nu_{12}=\epsilon_2-\epsilon_1$ and used detailed balance to derive Boltzmann's equation and the Planck distribution law. In contrast to Einstein's paper, Bohr's relation between frequency and energy difference is used in an essential way in Eddington's proof, If n_i is the expected number of atoms in state i, then the detailed balance between the processes of Eqs. (1) and (2) requires

$$n_1/n_2=(B_2{}^1/B_1{}^2)\{1+[A_2{}^1/B_2{}^1\rho(\nu_{12},T)]\}. \quad \text{(C.1)}$$

To derive the Boltzmann distribution, rather than to assume it, Eddington needed another equation. He considered a third energy level and the three detailed balance equations between these three pairs of levels. Using the identity

$$(n_1/n_2)(n_2/n_3)=n_1/n_3$$

and Eq. (C.1),

$$\frac{B_2{}^1}{B_1{}^2}\left(1+\frac{A_2{}^1}{B_2{}^1\rho(\nu_{12},T)}\right)\frac{B_3{}^2}{B_2{}^3}\left(1+\frac{A_3{}^2}{B_3{}^2\rho(\nu_{23},T)}\right)=\frac{B_3{}^1}{B_1{}^3}\left(1+\frac{A_3{}^1}{B_3{}^1\rho(\nu_{31},T)}\right). \quad \text{(C.2)}$$

As before, assume $\rho(\nu,T)$ becomes infinite as T becomes infinite. Then,

$$(B_2{}^1/B_1{}^2)(B_3{}^2/B_2{}^3)=B_3{}^1/B_1{}^3.$$

Using this relation, and inserting Wien's law, $\rho(\nu,T)=a\nu^3f(\nu/T)$, Eq. (C.2) becomes

$$\left(1+\frac{C_{12}}{f(\nu_{12}/T)}\right)\left(1+\frac{C_{23}}{f(\nu_{23}/T)}\right)=1+\frac{C_{13}}{f(\nu_{13}/T)}, \quad \text{(C.3)}$$

where $C_{12}=A_2{}^1/B_2{}^1\nu_{12}{}^3$.

Equation (C.3) must hold at all temperatures, and the functions C_{ij} are not temperature dependent. It may be rewritten in the form

$$a_1C_{12}{}^{-1}+a_2C_{23}{}^{-1}+a_3(C_{13}/C_{12}C_{23})=a_4, \quad \text{(C.4)}$$

where the coefficients a_i are functions of $f(\nu_{ij}/T)$. We can now obtain a set of three independent equations by writing Eq. (C.4) at three different temperatures. Solving for C_{12} produces an expression which is a combination of functions of ν_{23} and ν_{12}; it is not a function of any specific material parameters. Moreover, it is physically clear that C_{12} cannot in fact depend in any way on ν_{23}. Thus C_{12} is not a function of frequency at all and the C_{ij}'s are all equal to a universal constant C. Equation (C.3) can then be written,

$$\left(1+\frac{C}{f(\nu_{12}/T)}\right)\left(1+\frac{C}{f(\nu_{23}/T)}\right)=1+\frac{C}{f(\nu_{13}/T)}. \quad \text{(C.5)}$$

Since $\nu_{13}=\nu_{12}+\nu_{23}$, Eq. (C.5) takes the form

$$g(\nu_{12}/T)g(\nu_{23}/T)=g[(\nu_{12}+\nu_{23})/T]$$

and $g(\nu/T)=\exp(\gamma\nu/T)$. Then

$$f(\nu/T)=[\exp(\gamma\nu/T)-1]^{-1}$$

and

$$\rho(\nu,T)=\frac{C\nu^3}{[\exp(\gamma\nu/T)-1]}. \qquad (C.6)$$

This completes the derivation of Planck's law with C and γ to be evaluated by other means. The distribution function for the atoms can now be obtained from Eq. (C.1) by using both $A_{ij}/B_{ij}=C\nu_{ij}^3$, and the distribution function for radiation given in Eq. (C.6), Equation (C.1) then becomes

$$n_1/n_2=(B_2{}^1/B_1{}^2)\exp(\gamma\nu_{12}/T).$$

If g_1/g_2 is the asymptotic value of n_1/n_2 as T becomes infinite

$$n_1/n_2=(g_1/g_2)\exp(\gamma\nu_{12}/T). \qquad (C.7)$$

If we use Bohr's relation between frequency and energy, Eq. (C.7) is the Boltzmann formula. It has been derived here for either free or bound electrons but the weighting factors g_2 and g_1 have not been determined.

* A portion of this work was completed while the author was at RCA Laboratories, Princeton, New Jersey.

[1] A. Eddington, Phil. Mag. **50,** 803 (1925).

[2] A. Einstein, Phys. Z. **18,** 121 (1917).

[3] Undoubtedly the authors cited here have modified Einstein's argument for pedagogical reasons. The result remains a loss of some significant history. For example, the following quotation is taken from an article by W. E. Lamb, Jr., in *Quantum Optics and Electronics,* edited by DeWitt *et al.,* (Gordon and Breach, New York, 1964): "Using the Boltzmann distribution law for the atoms and the Planck distribution law for radiation density he (Einstein) derived the following important relation $A=4\hbar\omega^3/c^3B$." Other similar statements are contained in the following: Feynman Lectures in Physics, (Addison-Welsley, 1966) Vol. I, p. 42–48; O. S. Heavens, *Optical Masers* (John Wiley, N. Y. 1964). In addition, a correct description of the objective of the 1971 paper is given by Max Jammer, in *The Conceptual Development of Quantum Mechanics* (McGraw–Hill, New York, 1966). However, he makes an error which is important from the logical point of view. Jammer states that Einstein identified A/B and $\epsilon_j-\epsilon_i$ by comparing the expression for ρ obtained from detailed balance with Wien's *empirical* radiation law $\rho\alpha\nu^3\exp(-h\nu/kT)$ (an empirical formula) rather than Wien's law $\rho\alpha\nu^3 f(\nu/T)$. The most recent (1936) correct reference I have found is in Fowler, *Statistical Mechanics* (Macmillan, Cambridge, 1936), 2nd ed.

[4] M. Planck, *The Theory of Heat Radiation* (Dover, New York, 1959), pp. 69–86.

[5] W. Pauli, Z. Physik **18,** 72 (1923).

[6] A. Einstein and P. Ehrenfest, Z. Physik **19,** 301 (1923).

[7] P. Y. Kapitza and P. A. M. Dirac, Proc. Cambridge Phil. Soc. **29,** 231 (1933).

[8] H. Schwarz, Z. Physik **204,** 276 (1967).