M2-Condensed Matter and Nanophysics

Photon-Matter Interactions

Home work

We consider here a two-level atom interacting with a single mode of the electromagnetic field. When this mode is treated quantum mechanically, specific features occur in the atomic dynamics, such as damping and revivals of the Rabi oscillations.

1 Quantization of a Mode of the Electromagnetic Field

We recall that in *classical mechanics*, a harmonic oscillator of mass m and frequency $\omega/2\pi$ obeys the equations of motion $\mathrm{d}x/\mathrm{d}t = p/m$ and $\mathrm{d}p/\mathrm{d}t = -m\,\omega^2 x$ where x is the position and p the momentum of the oscillator. Defining the *reduced* variables $X(t) = x(t)\,\sqrt{m\omega/\hbar}$ and $P(t) = p(t)/\sqrt{\hbar m\omega}$, the equations of motion of the oscillator are

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \omega P \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = -\omega X \ , \tag{1}$$

and the total energy $U(t)$ is given by

$$U(t) = \frac{\hbar\omega}{2}(X^2(t) + P^2(t)) \ . \tag{2}$$

1.1. Consider a cavity for electromagnetic waves, of volume V. Throughout this chapter, we consider a single mode of the electromagnetic field, of the form

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{u}_x\, e(t) \sin kz \qquad \boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{u}_y\, b(t) \cos kz \ ,$$

where $\boldsymbol{u}_x$, $\boldsymbol{u}_y$ and $\boldsymbol{u}_z$ are an orthonormal basis. We recall Maxwell's equations in vacuum:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = 0 \qquad \boldsymbol{\nabla}\wedge\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \qquad \boldsymbol{\nabla}\wedge\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c^2}\frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

and the total energy $U(t)$ of the field in the cavity:

$$U(t) = \int_V \left(\frac{\epsilon_0}{2}E^2(\boldsymbol{r},t) + \frac{1}{2\mu_0}B^2(\boldsymbol{r},t)\right)\mathrm{d}^3r \quad \text{with } \epsilon_0\mu_0 c^2 = 1 \ . \tag{3}$$

(a) Express de/dt and db/dt in terms of $k, c, e(t), b(t)$.

(b) Express $U(t)$ in terms of $V, e(t), b(t), \epsilon_0, \mu_0$. One can take

$$\int_V \sin^2 kz \, \mathrm{d}^3r = \int_V \cos^2 kz \, \mathrm{d}^3r = \frac{V}{2} .$$

(c) Setting $\omega = ck$ and introducing the reduced variables

$$\chi(t) = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}}\, e(t) \qquad \Pi(t) = \sqrt{\frac{V}{2\mu_0 \hbar\omega}}\, b(t)$$

show that the equations for $d\chi/dt$, $d\Pi/dt$ and $U(t)$ in terms of χ, Π and ω are formally identical to equations 1) and 2).

1.2. The quantization of the mode of the electromagnetic field under consideration is performed in the same way as that of an ordinary harmonic oscillator. One associates to the physical quantities χ and Π, Hermitian operators $\hat{\chi}$ and $\hat{\Pi}$ which satisfy the commutation relation

$$[\hat{\chi}, \hat{\Pi}] = \mathrm{i} .$$

The Hamiltonian of the field in the cavity is

$$\hat{H}_{\mathrm{C}} = \frac{\hbar\omega}{2}\left(\hat{\chi}^2 + \hat{\Pi}^2\right) .$$

The energy of the field is quantized: $E_{\mathrm{n}} = (n + 1/2)\,\hbar\omega$ (n is a non-negative integer); one denotes by $|n\rangle$ the eigenstate of $\hat{H}_{\mathrm{C}}$ with eigenvalue E_n.

The *quantum states* of the field in the cavity are linear combinations of the set $\{|n\rangle\}$. The state $|0\rangle$, of energy $E_0 = \hbar\omega/2$, is called the "vacuum", and the state $|n\rangle$ of energy $E_n = E_0 + n\hbar\omega$ is called the "n photon state". A "photon" corresponds to an elementary excitation of the field, of energy $\hbar\omega$.

One introduces the "creation" and "annihilation" operators of a photon as $\hat{a}^\dagger = (\hat{\chi} - \mathrm{i}\hat{\Pi})/\sqrt{2}$ and $\hat{a} = (\hat{\chi} + \mathrm{i}\hat{\Pi})/\sqrt{2}$ respectively. These operators satisfy the usual relations:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \text{if} \quad n \neq 0 \qquad \text{and} \quad \hat{a}|0\rangle = 0 .$$

(a) Express $\hat{H}_{\mathrm{C}}$ in terms of $\hat{a}^\dagger$ and $\hat{a}$. The observable $\hat{N} = \hat{a}^\dagger\hat{a}$ is called the "number of photons".

The *observables* corresponding to the electric and magnetic fields at a point $\boldsymbol{r}$ are defined as:

$$\hat{\boldsymbol{E}}(\boldsymbol{r}) = \boldsymbol{u}_x \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\,(\hat{a} + \hat{a}^\dagger)\sin kz$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r}) = \mathrm{i}\boldsymbol{u}_y \sqrt{\frac{\mu_0\hbar\omega}{V}}\,(\hat{a}^\dagger - \hat{a})\cos kz .$$

The interpretation of the theory in terms of states and observables is the same as in ordinary quantum mechanics.

(b) Calculate the expectation values $\langle \boldsymbol{E}(\boldsymbol{r})\rangle$, $\langle \boldsymbol{B}(\boldsymbol{r})\rangle$, and $\langle n|\hat{H}_C|n\rangle$ in an n-photon state.

1.3. The following superposition:

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ , \tag{4}$$

where α is any complex number, is called a "quasi-classical" state of the field.

(a) Show that $|\alpha\rangle$ is a normalized eigenvector of the annihilation operator $\hat{a}$ and give the corresponding eigenvalue. Calculate the expectation value $\langle n\rangle$ of the number of photons in that state.

(b) Show that if, at time $t = 0$, the state of the field is $|\psi(0)\rangle = |\alpha\rangle$, then, at time t, $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|(\alpha \mathrm{e}^{-\mathrm{i}\omega t})\rangle$.

(c) Calculate the expectation values $\langle \boldsymbol{E}(\boldsymbol{r})\rangle_t$ and $\langle \boldsymbol{B}(\boldsymbol{r})\rangle_t$ at time t in a quasi-classical state for which α is real.

(d) Check that $\langle \boldsymbol{E}(\boldsymbol{r})\rangle_t$ and $\langle \boldsymbol{B}(\boldsymbol{r})\rangle_t$ satisfy Maxwell's equations.

(e) Calculate the energy of a *classical* field such that $\boldsymbol{E}_{cl}(\boldsymbol{r}, t) = \langle \boldsymbol{E}(\boldsymbol{r})\rangle_t$ and $\boldsymbol{B}_{\mathrm{cl}}(\boldsymbol{r}, t) = \langle \hat{\boldsymbol{B}}(\boldsymbol{r})\rangle_t$. Compare the result with the expectation value of $\hat{H}_C$ in the same quasi-classical state.

(f) Why do these results justify the name "quasi-classical" state for $|\alpha\rangle$ if $|\alpha| \gg 1$?

2 The Coupling of the Field with an Atom

Consider an atom at point $\boldsymbol{r}_0$ in the cavity. The motion of the center of mass of the atom in space is treated classically. Hereafter we restrict ourselves to the *two-dimensional* subspace of internal atomic states generated by the ground state $|f\rangle$ and an excited state $|e\rangle$. The origin of atomic energies is chosen in such a way that the energies of $|f\rangle$ and $|e\rangle$ are respectively $-\hbar\omega_A/2$ and $+\hbar\omega_A/2$ ($\omega_A > 0$). In the basis $\{|f\rangle, |e\rangle\}$, one can introduce the operators:

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\sigma}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

that is to say $\hat{\sigma}_+|f\rangle = |e\rangle$ and $\hat{\sigma}_-|e\rangle = |f\rangle$, and the atomic Hamiltonian can be written as: $\hat{H}_A = \frac{\hbar\omega_A}{2}\hat{\sigma}_z$.

The set of orthonormal states $\{|f, n\rangle, |e, n\rangle, n \geq 0\}$ where $|f, n\rangle \equiv |f\rangle \otimes |n\rangle$ and $|e, n\rangle \equiv |e\rangle \otimes |n\rangle$ forms a basis of the Hilbert space of the {atom+photons} states.

2.1. Check that it is an eigenbasis of $\hat{H}_0 = \hat{H}_A + \hat{H}_C$, and give the corresponding eigenvalues.

2.2. In *the remaining parts of the problem* we assume that the frequency of the cavity is exactly tuned to the Bohr frequency of the atom, i.e. $\omega = \omega_A$. Draw schematically the positions of the first 5 energy levels of $\hat{H}_0$. Show that, except for the ground state, the eigenstates of $\hat{H}_0$ are grouped in degenerate pairs.

2.3. The Hamiltonian of the electric dipole coupling between the atom and the field can be written as:

$$\hat{W} = \gamma \left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- \right) ,$$

where $\gamma = -d\sqrt{\hbar\omega/\epsilon_0 V} \sin kz_0$, and where the electric dipole moment d is determined experimentally.

(a) Write the action of $\hat{W}$ on the states $|f, n\rangle$ and $|e, n\rangle$.
(b) To which physical processes do $\hat{a}\hat{\sigma}_+$ and $\hat{a}^\dagger\hat{\sigma}_-$ correspond?

2.4. Determine the eigenstates of $\hat{H} = \hat{H}_0 + \hat{W}$ and the corresponding energies. Show that the problem reduces to the diagonalization of a set of 2×2 matrices. One hereafter sets:

$$|\phi_n^\pm\rangle = \frac{1}{\sqrt{2}}(|f, n+1\rangle \pm |e, n\rangle)$$

$$\frac{\hbar\Omega_0}{2} = \gamma = -d\sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin kz_0 \qquad \Omega_n = \Omega_0\sqrt{n+1} .$$

The energies corresponding to the eigenstates $|\phi_n^\pm\rangle$ are denoted $E_n^\pm$.

3 Interaction of the Atom and an "Empty" Cavity

In the following, one assumes that the atom crosses the cavity along a line where $\sin kz_0 = 1$.

An atom in the excited state $|e\rangle$ is sent into the cavity prepared in the vacuum state $|0\rangle$. At time $t = 0$, when the atom enters the cavity, the state of the system is $|e, n = 0\rangle$.

3.1. What is the state of the system at a later time t?

3.2. What is the probability $P_f(T)$ of finding the atom in the state f at time T when the atom leaves the cavity? Show that $P_f(T)$ is a periodic function of T (T is varied by changing the velocity of the atom).

3.3. The experiment has been performed on rubidium atoms for a couple of states (f, e) such that $d = 1.1 \times 10^{-26}$ C.m and $\omega/2\pi = 5.0 \times 10^{10}$ Hz. The volume of the cavity is 1.87×10^{-6} m^3 (we recall that $\epsilon_0 = 1/(36\pi 10^9)$ S.I.). The curve $P_f(T)$, together with the real part of its Fourier transform $J(\nu) = \int_0^\infty \cos(2\pi\nu T)\, P_f(T)\, dT$, are shown in Fig. 1. One observes a

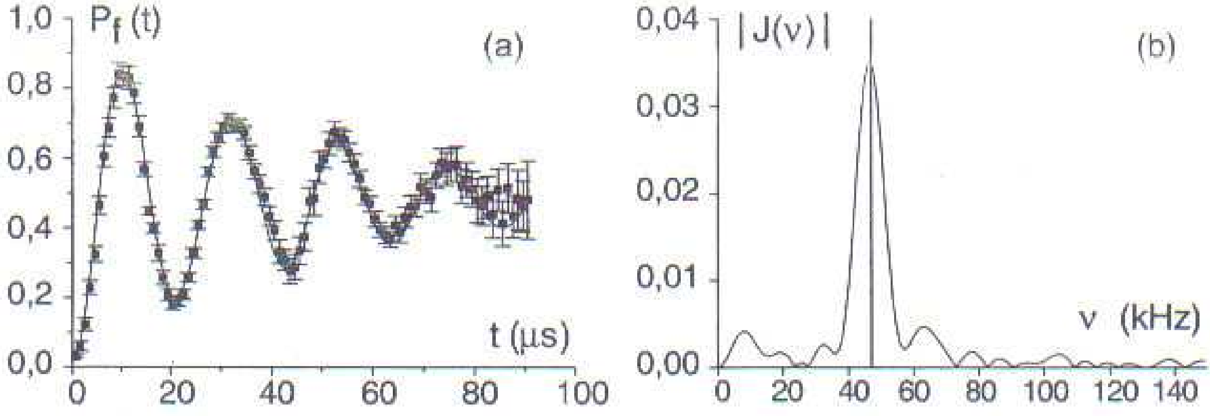

Fig. 1. **(a)** Probability $P_f(T)$ of detecting the atom in the ground state after it crosses a cavity containing zero photons; **(b)** Fourier transform of this probability, as defined in the text.

damped oscillation, the damping being due to imperfections in the experimental setup.

How do theory and experiment compare?

(We recall that the Fourier transform of a damped sinusoid in time exhibits a peak at the frequency of this sinusoid, whose width is proportional to the inverse of the characteristic damping time.)

4 Interaction of an Atom with a Quasi-Classical State

The atom, initially in the state $|e\rangle$, is now sent into a cavity where a quasi-classical state $|\alpha\rangle$ of the field has been prepared. At time $t = 0$ the atom enters the cavity and the state of the system is $|e\rangle \otimes |\alpha\rangle$.

4.1. Calculate the probability $P_f(T, n)$ of finding, at time T, the atom in the state $|f\rangle$ and the field in the state $|n+1\rangle$, for $n \geq 0$. What is the probability of finding the atom in the state $|f\rangle$ and the field in the state $|0\rangle$?

4.2. Write the probability $P_f(T)$ of finding the atom in the state $|f\rangle$, independently of the state of the field, as an infinite sum of oscillating functions.

4.3. On Fig. 2 are plotted an experimental measurement of $P_f(T)$ and the real part of its Fourier transform $J(\nu)$. The cavity used for this measurement is the same as in Fig. 1, but the field has been prepared in a quasi-classical state before the atom is sent in.

(a) Determine the three frequencies ν_0, ν_1, ν_2 which contribute most strongly to $P_f(T)$.
(b) Do the ratios ν_1/ν_0 and ν_2/ν_0 have the expected values?
(c) From the values $J(\nu_0)$ and $J(\nu_1)$, determine an approximate value for the mean number of photons $|\alpha|^2$ in the cavity.

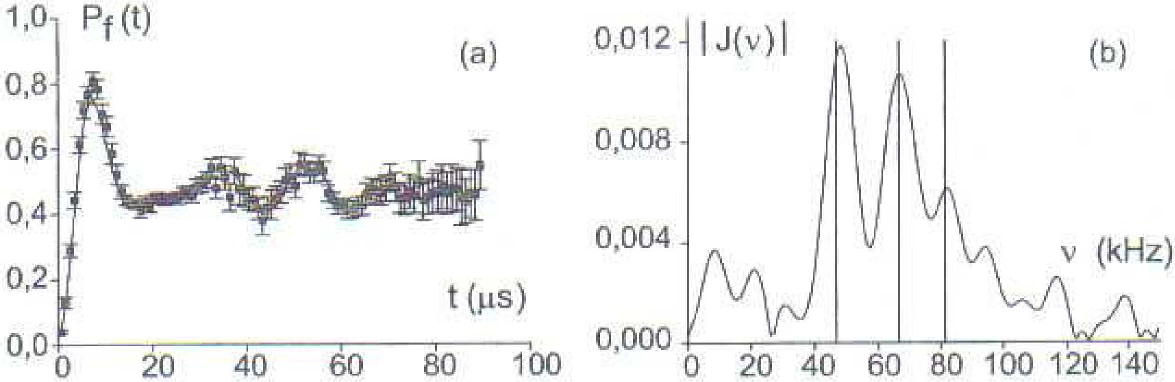

Fig. 2. (**a**) Probability $P_f(T)$ of measuring the atom in the ground state after the atom has passed through a cavity containing a quasi-classical state of the electromagnetic field; (**b**) Fourier transform of this probability.

5 Large Numbers of Photons: Damping and Revivals

Consider a quasi-classical state $|\alpha\rangle$ of the field corresponding to a large mean number of photons: $|\alpha|^2 \simeq n_0 \gg 1$, where n_0 is an integer. In this case, the probability $\pi(n)$ of finding n photons can be cast, in good approximation, in the form:

$$\pi(n) = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \simeq \frac{1}{\sqrt{2\pi n_0}} \exp\left(-\frac{(n-n_0)^2}{2\,n_0}\right) .$$

This Gaussian limit of the Poisson distribution can be obtained by using the Stirling formula $n! \sim n^n \mathrm{e}^{-n}\sqrt{2\pi n}$ and expanding $\ln \pi(n)$ in the vicinity of $n = n_0$.

5.1. Show that this probability takes significant values only if n is in a neighborhood δn of n_0. Give the relative value $\delta n / n_0$.

5.2. For such a quasi-classical state, one tries to evaluate the probability $P_f(T)$ of detecting the atom in the state f after its interaction with the field. In order to do this,

- one linearizes the dependence of Ω_n on n in the vicinity of n_0:

$$\Omega_n \simeq \Omega_{n_0} + \Omega_0 \frac{n - n_0}{2\sqrt{n_0+1}} , \qquad 5)$$

- one replaces the discrete summation in $P_f(T)$ by an integral.

(a) Show that, under these approximations, $P_f(T)$ is an oscillating function of T for short times, but that this oscillation is damped away after a characteristic time T_D. Give the value of T_D.
We recall that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(x-x_0)^2/2\sigma^2} \cos(\alpha x)\,\mathrm{d}x = \mathrm{e}^{-\alpha^2\sigma^2/2} \cos(\alpha x_0).$$

(b) Does this damping time depend on the mean value of the number of photons n_0?

(c) Give a qualitative explanation for this damping.

5.3. If one keeps the expression of $P_f(T)$ as a discrete sum, an exact numerical calculation shows that one expects a revival of the oscillations of $P_f(T)$ for certain times T_R large compared to T_D, as shown in Fig. 3. This phenomenon is called *quantum revival* and it is currently being studied experimentally.

Keeping the discrete sum but using the approximation (6.5), can you explain the revival qualitatively? How does the time of the first revival depend on n_0?

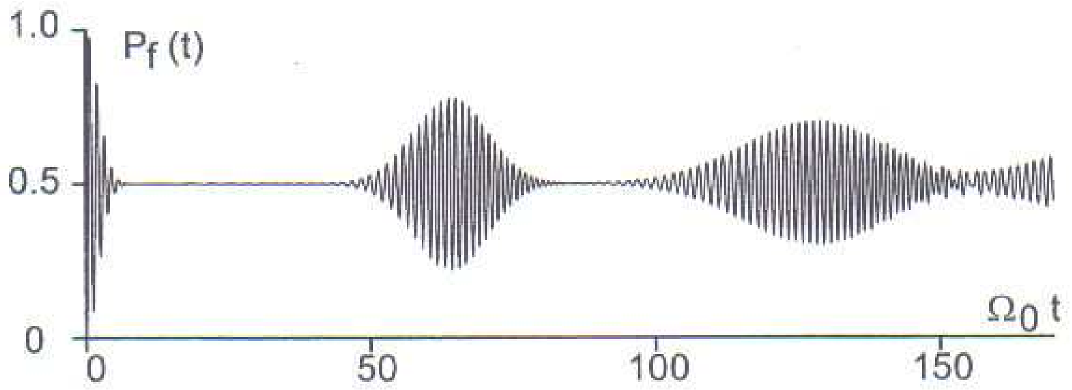

Fig. 3. Exact theoretical calculation of $P_f(T)$ for $\langle n \rangle \simeq 25$ photons.