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Chapter II

Classical and quantum descriptions of the electromagnetic field

1 Classical Electrodynamics

1.1 Maxwell-Lorentz Equations

1.1.1 Fundamental Equations

One has to distinguish two groups of physical quantities: (i) the fields with $\vec{E}(\vec{r},t)$ the electric field and $\vec{B}(\vec{r},t)$ the magnetic field; (ii) the "source" terms (sources of the field) with $\rho(\vec{r},t)$ the charge density and $\vec{j}(\vec{r},t)$ the courant density. The Maxwell equations read

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \frac{\rho}{\epsilon_0} && (1)\\ \vec{\nabla}.\vec{B} &= 0 && (2)\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} && (3)\\ \vec{\nabla}\times\vec{B} &= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\frac{1}{\epsilon_0 c^2}\vec{j}\ . && (4) \end{aligned}$$

These equations [(1) Poisson; (2) Gauss; (3) Faraday; (4) Ampère] for the fields are coupled to the equations that describe the charge dynamics (Lorentz equation)[1]

$$m_i\frac{d^2\vec{r}_i(t)}{dt^2} = q_i\left[\vec{E}(\vec{r}_i(t),t)+\vec{v}_i(t)\times\vec{B}(\vec{r}_i(t),t)\right]\ . \tag{5}$$

From the equations (1) and (4) one deduces the continuity equation or the charge conservation equation[2]

$$\frac{\partial\rho(\vec{r},t)}{\partial t} = -\vec{\nabla}.\vec{j}(\vec{r},t)\ , \tag{6}$$

with

$$\begin{aligned} \rho(\vec{r},t) &= \sum_i q_i\delta(\vec{r}-\vec{r}_i(t)) && (7)\\ \vec{j}(\vec{r},t) &= \sum_i q_i\vec{v}_i(t)\delta(\vec{r}-\vec{r}_i(t))\ . && (8) \end{aligned}$$

[1] This equation is not Lorentz invariant which is not the case for the Maxwell equations. The relativistic equation of motion reads: $\frac{d}{dt}\frac{m_i\vec{v}_i(t)}{\sqrt{1-\vec{v}_i^2(t)/c^2}} = q_i\left[\vec{E}(\vec{r}_i(t),t)+\vec{v}_i(t)\times\vec{B}(\vec{r}_i(t),t)\right]$. In the following, the charged particle dynamics will be considered within a non-relativistic framework.

[2] Remember that $\vec{\nabla}.(\vec{\nabla}\times\vec{B})=0$.

The state of the system "fields+charges" is completely determined at any time t if one knows $\left\{\vec{E}(\vec{r},t);\vec{B}(\vec{r},t);\vec{r}_i(t);\vec{v}_i(t)\right\}$. *Take care*, $\vec{r}$ is not a dynamical variable like $\vec{r}_i$ see 1.2.

According to the Noether's theorem (Emmy Noether March 23, 1882 - April 14, 1935 who worked with David Hilbert in Göttingen): the invariance of the Maxwell-Lorentz equations under a change of the time origin is related to the total energy conservation, H, which is a constant of motion (does not depend on time)

$$H = \sum_i \frac{1}{2} m_i v_i^2(t) + \frac{\epsilon_0}{2} \int \left(\|\vec{E}(\vec{r},t)\|^2 + c^2 \|\vec{B}(\vec{r},t)\|^2 \right) d\vec{r} \,. \tag{9}$$

The first term corresponds to the kinetic energy of the charged particles and the second one to the field energy. We have $\|\vec{E}(\vec{r},t)\|^2 \equiv \vec{E}.\vec{E}$. The Coulomb interaction among the charged particles is included in the field energy.

The invariance of the Maxwell-Lorentz equations with respect to a change of the origin of the spatial coordinate system is related to the conservation of the total linear momentum of the system, $\vec{P}$, which is a constant of motion

$$\vec{P} = \sum_i m_i \vec{v}_i(t) + \epsilon_0 \int \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)\; d\vec{r} \,. \tag{10}$$

Finally, The invariance of the Maxwell-Lorentz equations with respect to a change of the orientation of the axis of the spatial coordinate system is related to the conservation of the total angular momentum of the system, $\vec{J}$, which is a constant of motion

$$\vec{J} = \sum_i \vec{r}_i(t) \times m_i \vec{v}_i(t) + \epsilon_0 \int \vec{r} \times \left(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right)\; d\vec{r} \,. \tag{11}$$

1.1.2 Scalar and vector potentials, gauge transformation

From equations (2) and (3) one can write

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{12}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} U \,. \tag{13}$$

$\vec{A}(\vec{r},t)$ is the vector potential and $U(\vec{r},t)$ the scalar potential. One checks from equations (1) and (4) that these two quantities are solutions of[3]

$$\Box \vec{A} + \vec{\nabla} \left[\vec{\nabla}.\vec{A} + \frac{1}{c^2} \frac{\partial U}{\partial t} \right] = \frac{1}{\epsilon_0 c^2} \vec{j} \tag{14}$$

$$\Delta U = -\frac{\rho}{\epsilon_0} - \vec{\nabla}.\frac{\partial \vec{A}}{\partial t} \,. \tag{15}$$

[3]The d'Alembertian differential operator, $\Box$ reads, in a cartesian coordinate system $\Box = \frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$, where c is the speed of light, which can be rewritten in terms of the Laplacian operator as $\Box = \frac{\partial^2}{c^2 \partial t^2} - \Delta$.

In order to establish these equations we have used the differential relation $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)-\Delta\vec{A}$. $\vec{A}(\vec{r},t)$ and $U(\vec{r},t)$ are the "good fields". See also the *Generalized Stockes theorem.*

By definition, a gauge transformation is a transformation which keeps unchanged the fields $\vec{E}$ and $\vec{B}$. Let's consider the following transformations

$$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t)+\vec{\nabla}\phi(\vec{r},t) \tag{16}$$

$$U(\vec{r},t) = U_0(\vec{r},t)-\frac{\partial\phi(\vec{r},t)}{\partial t}\ . \tag{17}$$

Check that these transformations are gauge transformations (ex1).

The invariance of the Maxwell equations under a gauge transformation means that there is a flexibility in the choice of the scalar and vector potentials. The gauge can be fixed by imposing a condition on $\vec{A}$ and U. The two most important gauges (more than two exist) are the Lorentz's gauge and the Coulomb's gauge.

Lorentz's gauge: $\vec{\nabla}.\vec{A}+\frac{1}{c^2}\frac{\partial U}{\partial t}=0$. In this case, the Maxwell equations for the potentials read

$$\Box\vec{A}(\vec{r},t) = \frac{1}{\epsilon_0 c^2}\vec{j}(\vec{r},t) \tag{18}$$

$$\Box U(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t)\ . \tag{19}$$

If one uses covariant notations (in the spirit of special relativity) $A^\mu=\left\{\frac{U}{c},\vec{A}\right\};\partial_\mu=\left\{\frac{1}{c}\frac{\partial}{\partial t},\vec{\nabla}\right\};j^\mu=\left\{c\rho,\vec{j}\right\}$ the Lorentz's gauge condition reads $\sum_\mu\partial_\mu A^\mu=0\equiv\partial_\mu A^\mu$ (Einstein's convention) and the Maxwell equations for the potentials are $\sum_\nu\partial_\nu\partial^\nu A^\mu=\frac{1}{\epsilon_0c^2}j^\mu$. We have also $\partial^\nu=\sum_\mu g^{\mu\nu}\partial_\mu$ where $g^{\mu\nu}$ is the metric tensor given by $g^{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$.

Coulomb's gauge: $\vec{\nabla}.\vec{A}=0$. In this case, the Maxwell equations for the potentials read

$$\Box\vec{A}(\vec{r},t) = \frac{1}{\epsilon_0c^2}\vec{j}(\vec{r},t)-\frac{1}{c^2}\vec{\nabla}\frac{\partial U(\vec{r},t)}{\partial t} \tag{20}$$

$$\Delta U(\vec{r},t) = -\frac{1}{\epsilon_0}\rho(\vec{r},t)\ . \tag{21}$$

The last equation has the same form as the Poisson's equation in the case of the stationary regimes.

1.1.3 Maxwell equations in the Fourier space

Let

$$\vec{\mathcal{E}}(\vec{k},t)\equiv\frac{1}{\sqrt{8\pi^3}}\int\vec{E}(\vec{r},t)e^{-i\vec{k}.\vec{r}}d\vec{r} \tag{22}$$

be the Fourier transform (TF) of $\vec{E}(\vec{r},t)$. The inverse transformation (TF^{-1}) reads

$$\vec{E}(\vec{r},t) \equiv \frac{1}{\sqrt{8\pi^3}} \int \vec{\mathcal{E}}(\vec{k},t) e^{i\vec{k}.\vec{r}} d\vec{k} \; . \tag{23}$$

In the following we will use the notations:

- $\vec{E}(\vec{r},t) \longleftrightarrow \vec{\mathcal{E}}(\vec{k},t)$
- $\vec{B}(\vec{r},t) \longleftrightarrow \vec{\mathcal{B}}(\vec{k},t)$
- $\vec{A}(\vec{r},t) \longleftrightarrow \vec{\mathcal{A}}(\vec{k},t)$
- $\vec{A}_0(\vec{r},t) \longleftrightarrow \vec{\mathcal{A}}_0(\vec{k},t)$
- $U(\vec{r},t) \longleftrightarrow \bar{U}(\vec{k},t)$
- $U_0(\vec{r},t) \longleftrightarrow \bar{U}_0(\vec{k},t)$
- $\phi(\vec{r},t) \longleftrightarrow \bar{\phi}(\vec{k},t)$
- $\vec{j}(\vec{r},t) \longleftrightarrow \vec{\mathcal{J}}(\vec{k},t)$
- $\rho(\vec{r},t) \longleftrightarrow \bar{\rho}(\vec{k},t)$.

Proof the following (important) relations (ex2)

- $\vec{\nabla} \longleftrightarrow i\vec{k}$
- $\delta(\vec{r}-\vec{r}_i) \longleftrightarrow \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}.\vec{r}_i}$
- $\frac{1}{4\pi r} \longleftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2}$
- $\frac{\vec{r}}{4\pi r^3} \longleftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{-i\vec{k}}{k^2}$.

In the Fourier space the Maxwell equations (1),(2),(3), and (4) read

$$i\vec{k}.\vec{\mathcal{E}} = \frac{\bar{\rho}}{\epsilon_0} \tag{24}$$

$$i\vec{k}.\vec{\mathcal{B}} = 0 \tag{25}$$

$$i\vec{k} \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \tag{26}$$

$$i\vec{k} \times \vec{\mathcal{B}} = \frac{1}{c^2}\frac{\partial \vec{\mathcal{E}}}{\partial t} + \frac{1}{\epsilon_0 c^2}\vec{\mathcal{J}} \; . \tag{27}$$

The continuity equation (6) reads in the Fourier space

$$\frac{\partial \bar{\rho}(\vec{k},t)}{\partial t} = -i\vec{k}.\vec{\mathcal{J}}(\vec{k},t) \; . \tag{28}$$

For the potentials and the gauge transformation we have

$$\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}} \tag{29}$$

$$\vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{A}}}{\partial t} - i\vec{k}\bar{U} \ , \tag{30}$$

and

$$\vec{\mathcal{A}}(\vec{k},t) = \vec{\mathcal{A}}_0(\vec{k},t) + i\vec{k}\bar{\phi}(\vec{k},t) \tag{31}$$

$$\bar{U}(\vec{k},t) = \bar{U}_0(\vec{k},t) - \frac{\partial \bar{\phi}(\vec{k},t)}{\partial t} \ . \tag{32}$$

The time evolution equations of the potentials are

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k^2\right)\vec{\mathcal{A}} + i\vec{k}\left[i\vec{k}.\vec{\mathcal{A}} + \frac{1}{c^2}\frac{\partial \bar{U}}{\partial t}\right] = \frac{1}{\epsilon_0 c^2}\vec{\mathcal{J}} \tag{33}$$

$$k^2\bar{U} = -\frac{1}{\epsilon_0} - i\vec{k}.\frac{\partial \vec{\mathcal{A}}}{\partial t} \ . \tag{34}$$

1.1.4 Decomposition of the fields in terms of longitudinal and transverse components

By definition:

A longitudinal vector field $\vec{V}_{\parallel}(\vec{r},t)$ is a vector field such as $\vec{\nabla} \times \vec{V}_{\parallel}(\vec{r},t) = \vec{0}$ which leads in the Fourier space to $i\vec{k} \times \vec{\mathcal{V}}_{\parallel}(\vec{k},t) = \vec{0}$.

A transverse vector field $\vec{V}_{\perp}(\vec{r},t)$ is a vector field such as $\vec{\nabla}.\vec{V}_{\perp}(\vec{r},t) = 0$ which leads in the Fourier space to $i\vec{k}.\vec{\mathcal{V}}_{\perp}(\vec{k},t) = 0$.

Thus, in the Fourier space, a longitudinal field is parallel to $\vec{k}$ $(\forall \vec{k})$ and a transverse field is perpendicular to $\vec{k}$ $(\forall \vec{k})$.

Let's consider $\vec{\mathcal{V}}(\vec{k},t) = \vec{\mathcal{V}}_{\perp}(\vec{k},t) + \vec{\mathcal{V}}_{\parallel}(\vec{k},t)$ and $\hat{k} \equiv \vec{k}/k$ a unitary vector one can always write

$$\begin{cases} \vec{\mathcal{V}}_{\parallel}(\vec{k},t) = \hat{k}\left[\hat{k}.\vec{\mathcal{V}}(\vec{k},t)\right] \\ \vec{\mathcal{V}}_{\perp}(\vec{k},t) = \vec{\mathcal{V}}(\vec{k},t) - \vec{\mathcal{V}}_{\parallel}(\vec{k},t) \ . \end{cases} \tag{35}$$

Using the inverse Fourier transform on can obtain $\vec{V}_{\parallel}(\vec{r},t)$ and $\vec{V}_{\perp}(\vec{r},t)$.

From the equations (25)$i\vec{k}.\vec{\mathcal{B}} = 0$ and (24)$i\vec{k}.\vec{\mathcal{E}} = \frac{1}{\epsilon_0}\bar{\rho}(\vec{k},t)$ we obtain that $\vec{\mathcal{B}}_{\parallel}(\vec{k},t) = \vec{0}$ and $\vec{\mathcal{E}}_{\parallel}(\vec{k},t) = -\frac{i}{\epsilon_0}\bar{\rho}(\vec{k},t)\frac{\vec{k}}{k^2}$ (to be proven ex3).

Thus, the magnetic field is *purely transverse* ($\vec{\mathcal{B}}_{\parallel}(\vec{k},t) = \vec{0} \Rightarrow \vec{B}_{\parallel}(\vec{r},t) = \vec{0}$) and the longitudinal electric field is directly related to the charge distribution $\bar{\rho}(\vec{k},t)$. By using $\mathrm{TF}^{-1}\left\{-i\frac{\vec{k}}{k^2}\right\} = \frac{\sqrt{8\pi^3}}{4\pi}\frac{\vec{r}}{r^3}$ we obtain

$$\vec{E}_{\parallel}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}',t)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}d\vec{r}' \ , \tag{36}$$

and by using (7)

$$\vec{E}_{\parallel}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\vec{r}-\vec{r}_i(t)}{|\vec{r}-\vec{r}_i(t)|^3} \ . \tag{37}$$

Thus, the longitudinal electric field is the Coulomb field associated to the charge density $\rho(\vec{r},t)$. This result is independent of the gauge.

Be careful: For a point-like charge in $\vec{r}_i$ we have from the Poisson equation: $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} = \frac{\delta(\vec{r}-\vec{r}_i)}{\epsilon_0} q_i$. Thus $\vec{\nabla}.\vec{E} = 0$ everywhere except in $\vec{r}_i$ where the particle is localized. Therefore $\vec{E}$ is not a purely transverse field ! We have $i\vec{k}.\vec{\mathcal{E}} = \frac{q_i}{\epsilon_0}\frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}.\vec{r}_i} \neq 0$ everywhere.

Be careful: $\vec{r}$ is a continuous variable and $\vec{r}_i(t)$ is a discontinuous variable which represents the position of the particle i.

By decomposing equation (13) in the directions $\parallel$ and $\perp$ one obtains

$$\begin{cases} \vec{E}_{\parallel}(\vec{r},t) = -\frac{\partial \vec{A}_{\parallel}}{\partial t} - \vec{\nabla} U \\ \vec{E}_{\perp}(\vec{r},t) = -\frac{\partial \vec{A}_{\perp}}{\partial t} \ . \end{cases} \tag{38}$$

In the Coulomb gauge we have $\vec{A}_{\parallel} = \vec{0}$ leading to $\vec{E}_{\parallel} = -\vec{\nabla} U$. $U(\vec{r},t)$ is the Coulomb potential associated to the charge density $\rho(\vec{r},t)$ and is expressed as

$$U(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} d\vec{r}' \ . \tag{39}$$

This is the integral form of the Poisson equation (1) in the Coulomb gauge.

Proof that $\vec{A}_{\perp}$ is invariant under the gauge transformation (ex4).

Energy, linear and angular momenta

Energy: The energy can be decomposed in longitudinal and transverse parts:

$$H = \sum_i \frac{1}{2} m_i v_i^2(t) + H_{\parallel} + H_{\perp} \tag{40}$$

with

$$H_{\parallel} = \frac{\epsilon_0}{2}\int \left(\vec{E}_{\parallel}^2 + c^2 \vec{B}_{\parallel}^2\right) d\vec{r} \tag{41}$$

and

$$H_{\perp} = \frac{\epsilon_0}{2}\int \left(\vec{E}_{\perp}^2 + c^2 \vec{B}_{\perp}^2\right) d\vec{r} \ . \tag{42}$$

It can be proven by using the Parseval's relation (ex5)

$$\int \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) d\vec{r} = \int \vec{\mathcal{E}}^*(\vec{k},t).\vec{\mathcal{E}}(\vec{k},t) d\vec{k} \ . \tag{43}$$

It can be proven (ex6) that $H_{\parallel}$ can be expressed as

$$H_{\parallel} = \frac{1}{2\epsilon_0}\int \frac{\bar{\rho}^*(\vec{k},t)\bar{\rho}(\vec{k},t)}{k^2} d\vec{k} = \frac{1}{8\pi\epsilon_0}\int\int \frac{\rho(\vec{r},t)\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} d\vec{r} d\vec{r}' \ , \tag{44}$$

and finally as

$$H_{\parallel} = \sum_i \frac{q_i^2}{8\pi^3\epsilon_0} \int \frac{d\vec{k}}{k^2} + \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \; . \tag{45}$$

The first term represents the Coulomb energy of the particles (the Coulomb self-energy). This quantity diverges. The second term corresponds to the Coulomb interaction between the charged particles.

Linear momentum: From (10) one obtains

$$\vec{P} = \sum_i m_i \vec{v}_i + \vec{P}_{\parallel} + \vec{P}_{\perp} \tag{46}$$

with

$$\vec{P}_{\parallel,\perp} = \epsilon_0 \int \vec{E}_{\parallel,\perp}(\vec{r},t) \times \vec{B}_{\perp}(\vec{r},t) d\vec{r} \; . \tag{47}$$

It is straightforward to proof (ex7) that we have[4]

$$\vec{P}_{\parallel} = \int \rho(\vec{r},t)\vec{A}_{\perp}(\vec{r},t)d\vec{r} = \sum_i q_i \vec{A}_{\perp}(\vec{r}_i,t) \; . \tag{48}$$

We define the generalized linear momentum as: $\vec{p}_i \equiv m_i\vec{v}_i + q_i\vec{A}_{\perp}(\vec{r}_i)$ and thus we have $\vec{P} = \sum_i \vec{p}_i + \vec{P}_{\perp}$.

Angular momentum: From the definition (11) it is possible to show that

$$\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i + \vec{J}_{\perp} \tag{49}$$

with

$$\vec{J}_{\perp} = \epsilon_0 \int \vec{r} \times \vec{E}_{\perp}(\vec{r},t) \times \vec{B}_{\perp}(\vec{r},t) d\vec{r} \; . \tag{50}$$

Transverse equations: The equations (26) and (27) immediately lead to

$$\frac{\partial \vec{\mathcal{B}}_{\perp}(\vec{k},t)}{\partial t} = -i\vec{k} \times \vec{\mathcal{E}}_{\perp}(\vec{k},t) \tag{51}$$

$$\frac{\partial \vec{\mathcal{E}}_{\perp}(\vec{k},t)}{\partial t} = ic^2\vec{k} \times \vec{\mathcal{B}}_{\perp}(\vec{k},t) - \frac{1}{\epsilon_0}\vec{\mathcal{J}}_{\perp}(\vec{k},t) \; . \tag{52}$$

One can also show that

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k^2\right] \vec{\mathcal{A}}_{\perp}(\vec{k},t) = \frac{1}{\epsilon_0 c^2}\vec{\mathcal{J}}_{\perp}(\vec{k},t) \tag{53}$$

$$k^2 \bar{U}(\vec{k},t) = \frac{1}{\epsilon_0}\bar{\rho}(\vec{k},t) + i\vec{k}.\vec{\mathcal{A}}_{\parallel}(\vec{k},t) \; , \tag{54}$$

which leads in the real space to

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right] \vec{A}_{\perp}(\vec{r},t) = \frac{1}{\epsilon_0 c^2}\vec{j}_{\perp}(\vec{r},t) \; . \tag{55}$$

This is the propagation equation for the transverse fields.

[4] Use the formula $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$.

1.1.5 Analogy between the EM field and an ensemble of harmonic oscillators

The equations (52) are coupled. One looks for new variables which allows to decouple them (research of the eigenmodes) and to underline the analogy with the harmonic oscillator. If $\vec{j}_\perp(\vec{r},t) = \vec{0}$ we naturally have ($\omega = ck$)

$$\frac{\partial}{\partial t}\left[\vec{\mathcal{E}}_\perp(\vec{k},t) \pm c\hat{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)\right] = \pm i\omega\left[\vec{\mathcal{E}}_\perp(\vec{k},t) \pm c\hat{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)\right] . \tag{56}$$

Thus, one defines the new variables (normal variables) as

$$\vec{\alpha}(\vec{k},t) \equiv -i\sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\vec{\mathcal{E}}_\perp(\vec{k},t) - c\hat{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)\right] \tag{57}$$

$$\vec{\beta}(\vec{k},t) \equiv -i\sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\vec{\mathcal{E}}_\perp(\vec{k},t) + c\hat{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)\right] . \tag{58}$$

By inverting these equations one has

$$\vec{\mathcal{E}}_\perp(\vec{k},t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left[\vec{\alpha}(\vec{k},t) - \vec{\alpha}^*(-\vec{k},t)\right] \tag{59}$$

$$\vec{\mathcal{B}}_\perp(\vec{k},t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\hat{k}\times\left[\vec{\alpha}(\vec{k},t) + \vec{\alpha}^*(-\vec{k},t)\right]/c . \tag{60}$$

By including the source term $\vec{j}_\perp(\vec{r},t)$, the equation (56) becomes

$$\frac{\partial\vec{\alpha}(\vec{k},t)}{\partial t} + i\omega\vec{\alpha}(\vec{k},t) = \frac{i}{\sqrt{2\hbar\omega\epsilon_0}}\vec{\mathcal{J}}_\perp(\vec{k},t) . \tag{61}$$

Here a few comments must be given:

- $\vec{\alpha}(\vec{k},t)$ is a transverse quantity according to its definition (57)
- The last equation is similar to the one of a fictive oscillator with a frequency ω under the action of the term $\vec{\mathcal{J}}_\perp(\vec{k},t)$ (to be proven ex8).
- $\vec{\beta}(\vec{k},t) = -\vec{\alpha}^*(-\vec{k},t)$
- The constant $\sqrt{\frac{\epsilon_0}{2\hbar\omega}}$ is introduced for the purpose of the future quantization of the model.

Expression of the energy: By using the definitions (60) and (42) one shows that (to be done ex9)

$$H_\perp = \int d\vec{k}\sum_\epsilon \frac{\hbar\omega}{2}\left(\alpha_\epsilon(\vec{k},t)\alpha_\epsilon^*(\vec{k},t) + \alpha_\epsilon^*(\vec{k},t)\alpha_\epsilon(\vec{k},t)\right) \tag{62}$$

where ϵ are the components of a unitary vector perpendicular to the direction $\vec{k}$. One has $\vec{\alpha}(\vec{k},t) \equiv \alpha_\epsilon(\vec{k},t)\hat{\epsilon} + \alpha_{\epsilon'}(\vec{k},t)\hat{\epsilon}'$ and $\hat{\epsilon}.\hat{k} = \hat{\epsilon}'.\hat{k} = \hat{\epsilon}.\hat{\epsilon}' = 0$. The unitary vectors $\hat{\epsilon}$ and $\hat{\epsilon}'$ can be complex quantities in the case of circular polarizations.

Linear momentum

$$\vec{P}_\perp = \int d\vec{k} \sum_\epsilon \frac{\hbar\vec{k}}{2} \left(\alpha_\epsilon(\vec{k},t)\alpha_\epsilon^*(\vec{k},t) + \alpha_\epsilon^*(\vec{k},t)\alpha_\epsilon(\vec{k},t) \right) . \quad (63)$$

Let's now consider the expressions of the fields in the real space. We have

$$\vec{E}_\perp(\vec{r},t) = i \int d\vec{k} \sum_\epsilon \sqrt{\frac{\hbar\omega}{2\epsilon_0 8\pi^3}} \left(\alpha_\epsilon(\vec{k},t)\hat{\epsilon}e^{i\vec{k}.\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\hat{\epsilon}^* e^{-i\vec{k}.\vec{r}} \right) \quad (64)$$

$$\vec{B}_\perp(\vec{r},t) = i \int d\vec{k} \sum_\epsilon \frac{1}{c} \sqrt{\frac{\hbar\omega}{2\epsilon_0 8\pi^3}} \left(\alpha_\epsilon(\vec{k},t)\hat{k}\times\hat{\epsilon}e^{i\vec{k}.\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\hat{k}\times\hat{\epsilon}^* e^{-i\vec{k}.\vec{r}} \right) . \quad (65)$$

Vector potential
As $\vec{E}_\perp(\vec{r},t) = -\frac{\partial \vec{A}_\perp(\vec{r},t)}{\partial t}$ and $\vec{B}_\perp(\vec{r},t) = \vec{\nabla}\times\vec{A}_\perp(\vec{r},t)$ one obtains

$$\vec{A}_\perp(\vec{r},t) = \int d\vec{k} \sum_\epsilon \frac{1}{kc} \sqrt{\frac{\hbar\omega}{2\epsilon_0 8\pi^3}} \left(\alpha_\epsilon(\vec{k},t)\hat{\epsilon}e^{i\vec{k}.\vec{r}} + \alpha_\epsilon^*(\vec{k},t)\hat{\epsilon}^* e^{-i\vec{k}.\vec{r}} \right) \quad (66)$$

$$\vec{\mathcal{A}}_\perp(\vec{k},t) = \sqrt{\frac{\hbar}{2\epsilon_0}} \left(\vec{\alpha}(\vec{k},t) + \vec{\alpha}^*(-\vec{k},t) \right) \quad (67)$$

$$\vec{\alpha}(\vec{k},t) = \left(\omega\vec{\mathcal{A}}_\perp(\vec{k},t) - i\vec{\mathcal{E}}_\perp(\vec{k},t) \right) . \quad (68)$$

For the free fields $\vec{j}_\perp = \vec{0}$ the equation of motion of $\alpha_\epsilon(\vec{k},t)$ gives [solution of the equation (61) with $\vec{\mathcal{J}}_\perp = \vec{0}$]

$$\alpha_\epsilon(\vec{k},t) = \alpha_\epsilon(\vec{k})e^{-i\omega t} . \quad (69)$$

Thus, the free field is a superposition of plane waves $e^{i(\vec{k}.\vec{r}-\omega t)}$ or, more explicitly

$$\vec{E}_\perp(\vec{r},t) = i \int d\vec{k} \sum_\epsilon \sqrt{\frac{\hbar\omega}{2\epsilon_0 8\pi^3}} \left(\alpha_\epsilon(\vec{k})\hat{\epsilon}e^{i(\vec{k}.\vec{r}-\omega t)} - \alpha_\epsilon^*(\vec{k})\hat{\epsilon}^* e^{-i(\vec{k}.\vec{r}-\omega t)} \right) . \quad (70)$$

1.1.6 Decomposition of the fields in eigenmodes: EM field in a cavity

Let's consider that the field is inside a cavity of size L (cube of volume L^3) with periodic boundary conditions. When the cavity tends to infinity one can recover the field in all space.

For the sake of simplicity, let's suppose that $\vec{j}_\perp = \vec{0}$ (free field)

We have periodic boundary conditions $k_{x,y,z} = 2\pi n_{x,y,z}/L$. The variables $\alpha_\epsilon(\vec{k},t)$ are replaced by the discrete variables $\alpha_{\vec{k}_i\epsilon_i}(t)$. In order to simplify the notation, we will use in the following $\alpha_i \to \left(\frac{2\pi}{L}\right)^{3/2} \alpha_{\vec{k}_i\epsilon_i}$ where the index i expresses the set $(\vec{k}_i, \epsilon_i)$. We also have

$$\int d\vec{k} \sum_\epsilon f(\vec{k},\epsilon) \to \sum_i \left(\frac{2\pi}{L}\right)^3 f(\vec{k}_i,\epsilon_i) . \quad (71)$$

We get

$$H_\perp = \sum_i \frac{\hbar\omega_i}{2}\left(\alpha_i(t)\alpha_i^*(t) + \alpha_i^*(t)\alpha_i(t)\right) \tag{72}$$

$$\vec{P}_\perp = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\alpha_i(t)\alpha_i^*(t) + \alpha_i^*(t)\alpha_i(t)\right) \tag{73}$$

$$\vec{E}_\perp(\vec{r},t) = i\sum_i e_{\omega_i}\left(\alpha_i(t)\hat{\epsilon}_i e^{i\vec{k}_i.\vec{r}} - \alpha_i^*(t)\hat{\epsilon}_i^* e^{-i\vec{k}_i.\vec{r}}\right) \tag{74}$$

$$\vec{B}_\perp(\vec{r},t) = i\sum_i b_{\omega_i}\left(\alpha_i(t)\hat{k}_i\times\hat{\epsilon}_i e^{i\vec{k}_i.\vec{r}} - \alpha_i^*(t)\hat{k}_i\times\hat{\epsilon}_i^* e^{-i\vec{k}_i.\vec{r}}\right) \tag{75}$$

$$\vec{A}_\perp(\vec{r},t) = \sum_i a_{\omega_i}\left(\alpha_i(t)\hat{\epsilon}_i e^{i\vec{k}_i.\vec{r}} + \alpha_i^*(t)\hat{\epsilon}_i^* e^{-i\vec{k}_i.\vec{r}}\right) \tag{76}$$

with $a_{\omega_i} = e_{\omega_i}/\omega_i$, $b_{\omega_i} = e_{\omega_i}/c$ and $e_{\omega_i} = \sqrt{\frac{\hbar\omega_i}{2\epsilon_0 L^3}}$. We also have

$$\dot{\alpha}_i + i\omega_i\alpha_i = \frac{i}{(2\epsilon_0\hbar\omega_i)^{1/2}} j_i\ , \tag{77}$$

with

$$j_i(t) = \frac{1}{\sqrt{L^3}}\int d\vec{r} e^{-i\vec{k}_i.\vec{r}}\vec{\epsilon}_i.\vec{j}(\vec{r},t)\ . \tag{78}$$

It is natural to expand the vector potential $\vec{A}$ in Fourier series as follows

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}} \vec{\mathcal{A}}_{\vec{k}}(t)e^{i\vec{k}.\vec{r}} + \vec{\mathcal{A}}_{\vec{k}}^*(t)e^{-i\vec{k}.\vec{r}} \tag{79}$$

with $\vec{k}$ such that ($k_i = 2\pi n_i/L$) with n_i relative integer. Starting from the transverse vector potential $\vec{A}_\perp$ which obeys the equation

$$\vec{\nabla}^2\vec{A}_\perp(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}_\perp}{\partial t^2} = 0 \tag{80}$$

one gets the following equation

$$\frac{\partial^2\vec{\mathcal{A}}_{\vec{k}}}{\partial t^2} + \omega_k^2\vec{\mathcal{A}}_{\vec{k}} = 0 \tag{81}$$

with $\omega_k = ck$. Its solution is

$$\vec{\mathcal{A}}_{\vec{k}}(t) = \vec{\mathcal{A}}_{\vec{k}}e^{-i\omega_k t}\ . \tag{82}$$

From $\vec{\mathcal{B}} = i\vec{k}\times\vec{\mathcal{A}}$ and $\frac{\partial\vec{\mathcal{E}}_\perp(\vec{k},t)}{\partial t} = ic^2\vec{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)$ one gets

$$\vec{E}_{\vec{k}} = i\omega_k\left[\vec{\mathcal{A}}_{\vec{k}}e^{-i(\omega_k t-\vec{k}.\vec{r})} - \vec{\mathcal{A}}_{\vec{k}}^*e^{i(\omega_k t-\vec{k}.\vec{r})}\right] \tag{83}$$

$$\vec{B}_{\vec{k}} = i\vec{k}\times\left[\vec{\mathcal{A}}_{\vec{k}}e^{-i(\omega_k t-\vec{k}.\vec{r})} + \vec{\mathcal{A}}_{\vec{k}}^*e^{i(\omega_k t-\vec{k}.\vec{r})}\right]\ . \tag{84}$$

The mode energy $\vec{k}$ is given by

$$H_{\vec{k}} = \frac{\epsilon_0}{2} \int_{cav} \left(\vec{E}^2_{\vec{k}} + c^2 \vec{B}^2_{\vec{k}} \right) d\vec{r} \quad (85)$$

which leads to $H_{\vec{k}} = 2\epsilon_0 \omega_k^2 V \vec{\mathcal{A}}_{\vec{k}}.\vec{\mathcal{A}}^*_{\vec{k}}$.

One can redefine two coordinates $Q_{\vec{k}}$ et $P_{\vec{k}}$ as

$$\vec{\mathcal{A}}_{\vec{k}} = \frac{1}{\sqrt{4\epsilon_0 V \omega_k^2}} \left(\omega_k Q_{\vec{k}} + i P_{\vec{k}} \right) \hat{\epsilon}_{\vec{k}} \quad (86)$$

$$\vec{\mathcal{A}}^*_{\vec{k}} = \frac{1}{\sqrt{4\epsilon_0 V \omega_k^2}} \left(\omega_k Q_{\vec{k}} - i P_{\vec{k}} \right) \hat{\epsilon}^*_{\vec{k}} \ . \quad (87)$$

which leads to $H_{\vec{k}} = \frac{1}{2}\left(P^2_{\vec{k}} + \omega_k^2 Q^2_{\vec{k}} \right)$. One recognizes the energy of one harmonic oscillator having a spatial coordinate $Q_{\vec{k}}$ and its associated canonical conjugate momentum $P_{\vec{k}}$.

At this stage, it is tempting to immediately quantize the field by using the hamiltonian associated to the harmonic oscillator. However one must be more careful...
[5]

1.2 Hamiltonian formulation of the classical EM field

Before applying the usual canonical quantization rules of QM to the EM field, it is necessary to be sure that the expressions that we have derived so far can be obtained from a Lagrangian or a Hamiltonian theory. Then, and only then one could apply the principles of quantum mechanics. The Lagrangian formulation of the classical electrodynamics from the principle of least action was done by Karl Schwarzschild in 1903.

1.2.1 variational principle

A) Discrete number of degrees of freedom

[5] *The problem of the quantization of classical electrodynamics*

(i) Elementary approach

The total system is equivalent to a set of particles and oscillators interacting together. For the particles, the couples $(\vec{r}_\alpha, \vec{p}_\alpha)$ become quantum operators $(\hat{\vec{r}}_\alpha, \hat{\vec{p}}_\alpha)$ which obey the commutation relations $[\hat{r}_{\alpha i}, \hat{r}_{\beta j}] = [\hat{p}_{\alpha i}, \hat{p}_{\beta j}] = 0$ et $[\hat{r}_{\alpha i}, \hat{p}_{\beta j}] = i\hbar \delta_{\alpha\beta}\delta_{ij}$. For the fields, the couples (α_i, α_i^*) of the oscillator i (mode i) become the quantum operators $(\hat{a}_i, \hat{a}_i^+)$ with $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0$ and $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$.

This is not sufficient since this quantization procedure is not based on the Lagrangian or the Hamiltonian formulation of the classical electrodynamics.

(ii) Rigorous approach

Firstly, one shows that the Maxwell-Lorentz equations are equivalent to the Lagrange equations of a certain Lagrangian. Then, the canonical quantization consists in associating each couple formed by a generalized coordinate and its canonically conjugate momentum to two operators having a commutator equals to $i\hbar$.

Comment concerning the approach (i): We do not know "a priori" if r_α and p_α on the one hand and α_i and α_i^* on the other hand can be considered as conjugate dynamical variables.

Let S be the classical action of a one dimensional system (1D)

$$S = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt \; . \tag{88}$$

One has two dynamical variables x and $\dot{x}$ and the phase space has two dimensions. Let M_1 the initial state and M_2 the final state of the system. From the principle of least action, the real path (the physical one) is the one for which the action is extremal.

For a system having N degrees of freedom, if one gives the N generalized coordinates $x_1, ...x_N$ and the associated velocities $\dot{x}_1, ...\dot{x}_N$ at a given time, the motion of the system at any ulterior time is completely determined.

$(x_j, \dot{x}_j)$ with $j = 1, ...N$ correspond to $2N$ dynamical variables.

If $L'(x_j, \dot{x}_j, t) = L(x_j, \dot{x}_j, t) + \frac{d}{dt} f(x_j, t)$ then $S' = \int_{t_1}^{t_2} L'dt = S + f(x_j(t_2), t_2) - f(x_j(t_1), t_1)$. As the initial and final positions are fixed, S and S' differ from each other only by a constant. Consequently they have the same extremum.

Conjugate Moments

By definition we have $p_j \equiv \frac{\partial L}{\partial \dot{x}_j}$. By using the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_j} \right) = \frac{\partial L}{\partial x_j} \tag{89}$$

we see that $\dot{p}_j = \frac{\partial L}{\partial x_j}$.

Hamiltonian

By definition we have

$$H(x_j, p_j) = \sum_j \dot{x}_j p_j - L \; . \tag{90}$$

Hamilton's equations

$$\begin{cases} \dot{x}_j = \frac{\partial H}{\partial p_j} \\ \dot{p}_j = -\frac{\partial H}{\partial x_j} \end{cases} \tag{91}$$

Comment: For a system with N degrees of freedom there are N Lagrange's equations (second order differential equations) and $2N$ Hamilton's equations (first order differential equations).

Generalized complex coordinates

Let $X = \frac{1}{\sqrt{2}}(x_1 + ix_2)$ a complex variable (x_1 and x_2 are for example two coordinates). Instead of writing the lagrangian as $L(x_1, x_2, \dot{x}_1, \dot{x}_2, t)$ one may write it as $L(X, X^*, \dot{X}, \dot{X}^*, t)$. We have 4 dynamical variables: X, $\dot{X}$, X^* and $\dot{X}^*$. The new Lagrange equations read

$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{X}} = \frac{\partial L}{\partial X} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{X}^*} = \frac{\partial L}{\partial X^*} \end{cases} \tag{92}$$

Similarly to the real case one defines the conjugate momentum as $P = \frac{1}{\sqrt{2}}(p_1 + ip_2)$ and $P \equiv \left(\frac{\partial L}{\partial \dot{X}}\right)^* = \frac{\partial L}{\partial \dot{X}^*}$. The last equality comes from the fact that L is a real quantity. Please check all these relations for $L = T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$ (ex10).

Quantization

In order to quantize the model one replaces the conjugate variables (in the canonical sense) by quantum operators that verify the following commutation relations

$$\begin{cases} [\hat{x}_n, \hat{x}_m] = 0 \\ [\hat{p}_n, \hat{p}_m] = 0 \\ [\hat{x}_n, \hat{p}_m] = i\hbar\delta_{nm} \ . \end{cases} \tag{93}$$

The hats denote operators. For the generalized complex coordinates we have $(\hat{A} \to A_{ij}; \hat{A}^+ \to^t \bar{A} \to A^*_{ji})$

$$\begin{cases} \left[\hat{X}, \hat{X}^+\right] = 0 \\ \left[\hat{P}, \hat{P}^+\right] = 0 \\ \left[\hat{X}, \hat{P}\right] = 0 \\ \left[\hat{X}^+, \hat{P}^+\right] = 0 \\ \left[\hat{X}, \hat{P}^+\right] = i\hbar \\ \left[\hat{X}^+, \hat{P}\right] = i\hbar \end{cases} \tag{94}$$

$\hat{X}^+$ denotes the hermitic conjugate of the operator $\hat{X}$. For example, one has $\hat{P}^+ = \frac{1}{\sqrt{2}}(\hat{p}_1 - i\hat{p}_2)$ since $\hat{p}_1$ and $\hat{p}_2$ are hermitic operators. Proof the above relations (ex11).

B) Continuous number of degrees of freedom

The most beautiful example of the passage between discrete and continuous is the electromagnetic field that exists everywhere in space. Thus, we need a continuous number of dynamical variables for describing the field. Let $A_j(\vec{r},t)$ be the components of the vector potential. $\vec{r}$ **is a continuous index** and is not a dynamical variable ! The indices j is discrete and represents, for example, the components (x,y,z). Similarly to the discrete case, $A_j(\vec{r},t)$ and the velocities $\dot{A}_j(\vec{r},t)$ constitute a set of dynamical variables for the system. Ones defines the lagrangian density $\mathcal{L}(A_j,\dot{A}_j,\partial_i A_j,t)$ and the corresponding action S

$$S=\int_{t_1}^{t_2} dt \int \mathcal{L}(A_j,\dot{A}_j,\partial_i A_j,t) d\vec{r} \,. \tag{95}$$

The term $\partial_i A_j$ is necessary for describing the Maxwell equations that are non-local (the evolution of the coordinate $A_j(\vec{r},t)$ is coupled with the evolution of the coordinate of a neighbor point). If one apply the principle of least action to (95) one obtains the Lagrange equations for the fields

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{A}_j}=\frac{\partial \mathcal{L}}{\partial A_j}-\sum_{i=x,y,z}\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)} \,. \tag{96}$$

Conjugate moments

They are defined as (by analogy with the discrete case)

$$\Pi_j(\vec{r}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_j(\vec{r})} \,. \tag{97}$$

Quantization

We have the following quantization rules

$$\left\{ \begin{array}{ll} \left[\hat{A}_i(\vec{r}),\hat{A}_j(\vec{r}')\right] &= 0 \\ \left[\hat{\Pi}_i(\vec{r}),\hat{\Pi}_j(\vec{r}')\right] &= 0 \\ \left[\hat{A}_i(\vec{r}),\hat{\Pi}_j(\vec{r}')\right] &= i\hbar\delta_{ij}\delta(\vec{r}-\vec{r}') \,. \end{array} \right. \tag{98}$$

The operators $\hat{A}_i(\vec{r})$ et $\hat{\Pi}_j(\vec{r}')$ are *field operators*.

Complex fields

We are now working with complex fields as for example in the reciprocal space of the electromagnetism or with the Schrödinger field $\Psi(\vec{r},t)$...

We have

$$L=\int d\vec{k}\mathcal{L}(\mathcal{A}_j(\vec{k}),\dot{\mathcal{A}}_j(\vec{k}),\partial_i\mathcal{A}_j(\vec{k}),\mathcal{A}_j^*(\vec{k}),\dot{\mathcal{A}}_j^*(\vec{k}),\partial_i\mathcal{A}_j^*(\vec{k})) \,, \tag{99}$$

with $\partial_i \to (\partial/\partial k_x, \partial/\partial k_y, \partial/\partial k_z)$.

The new Lagrange equations read

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathcal{A}}_j} &= \frac{\partial\mathcal{L}}{\partial\mathcal{A}_j} - \sum_{i=k_x,k_y,k_z}\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\mathcal{A}_j)} \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathcal{A}}^*_j} &= \frac{\partial\mathcal{L}}{\partial\mathcal{A}^*_j} - \sum_{i=k_x,k_y,k_z}\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\mathcal{A}^*_j)} \ .
\end{aligned} \tag{100}$$

The associated moment (conjugate) to $\mathcal{A}_j(\vec{k})$ is

$$\pi_j(\vec{k}) \equiv \left(\frac{\partial\mathcal{L}}{\partial\dot{\mathcal{A}}_j(\vec{k})}\right)^* = \frac{\partial\mathcal{L}}{\partial\dot{\mathcal{A}}^*_j(\vec{k})} \ . \tag{101}$$

The associated moment to $\mathcal{A}^*_j(\vec{k})$ is $\pi^*_j(\vec{k})$.

Quantization

We have the following quantization rules

$$\left\{\begin{array}{l}
\left[\hat{\mathcal{A}}_i(\vec{k}),\hat{\mathcal{A}}_j(\vec{k}')\right] = 0 \\
\left[\hat{\pi}_i(\vec{k}),\hat{\pi}_j(\vec{k}')\right] = 0 \\
\left[\hat{\mathcal{A}}_i(\vec{k}),\hat{\pi}_j(\vec{k}')\right] = 0 \\
\left[\hat{\mathcal{A}}_i(\vec{k}),\hat{\pi}^+_j(\vec{k}')\right] = i\hbar\delta_{ij}\delta(\vec{k}-\vec{k}') \ .
\end{array}\right. \tag{102}$$

1.2.2 Non-relativistic Lagrangian of the system "fields-particles"

We start from the Lagrangian $L = L_P + L_R + L_I$

$$L \equiv \sum_i \frac{1}{2}m_i\dot{\vec{r}}_i^{\,2} + \frac{\epsilon_0}{2}\int\left(\vec{E}^2 + c^2\vec{B}^2\right)d\vec{r} + \sum_i q_i\dot{\vec{r}}_i.\vec{A}(\vec{r}_i) - q_iU(\vec{r}_i) \ , \tag{103}$$

with $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}U$. In the following we check that the Lagrangian L leads to the Maxwell-Lorentz equations. The interaction Lagrangian can be written as

$$L_I \equiv \int\left[\vec{j}(\vec{r}).\vec{A}(\vec{r}) - \rho(\vec{r})U(\vec{r})\right]d\vec{r} \equiv \int\mathcal{L}_I d\vec{r} \ , \tag{104}$$

where $\mathcal{L}_I$ denotes the Lagrangian density.

In (103) the generalized variables are: $\left\{(\vec{r}_i)_j;(\dot{\vec{r}}_i)_j\right\}$, $\left\{\vec{A}_j(\vec{r},t);\dot{\vec{A}}_j(\vec{r},t)\right\}$ and $\left\{U(\vec{r},t);\dot{U}(\vec{r},t)\right\}$. The last two brackets denote the dynamical variables of the field. Let's remind that the index j denotes the vectors components.

Gauge Invariance

1) The field Lagrangian L_R is not modified since it only involves the fields $\vec{E}$ and $\vec{B}$ and by definition, a gauge transformation does not change the fields $\vec{E}$ et $\vec{B}$.

2) The Lagrangian of the particles L_P is not modified.

3) Only the interaction Lagrangian is modified.

Indeed we have $L_I = \int \left[\vec{j}(\vec{r}).\vec{A}(\vec{r}) - \rho(\vec{r})U(\vec{r})\right] d\vec{r} = \int \mathcal{L}_I d\vec{r}$ is transformed under the gauge transformations (17) as $\mathcal{L}'_I = \mathcal{L}_I + \mathcal{L}_1$ with

$$\mathcal{L}_1 = \vec{j}.\vec{\nabla}\phi + \rho\frac{\partial\phi}{\partial t} = \vec{\nabla}.(\vec{j}\phi) + \frac{\partial}{\partial t}(\rho\phi) - \left(\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t}\right)\phi\,. \tag{105}$$

We have used the equality $\vec{\nabla}.(\vec{j}\phi) = (\vec{\nabla}.\vec{j})\phi + \vec{j}.\vec{\nabla}\phi$.

- If one integrates over the whole space, the term with the divergence vanishes (ex12).

- A time derivative does not change the equations of motion.

Finally, the charge conservation $\left(\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t}\right) = 0$ is a necessary condition for the gauge invariance. This is in agreement with the Noether's theorem.

Let us now show that the Lagrangian (103) leads to the Maxwell-Lorentz equations.

Lorentz equation

From the Lagrange equations for the particles

$$\frac{d}{dt}\frac{\partial L}{\partial(\dot{\vec{r}}_i)_j} = \frac{\partial L}{\partial(\vec{r}_i)_j} \tag{106}$$

ones obtains the Lorentz equation it means (ex13)[6]

$$m_i\frac{d^2\vec{r}_i}{dt^2} = q_i\vec{E}(\vec{r}_i) + q_i\dot{\vec{r}}_i \times \vec{B}(\vec{r}_i)\,. \tag{107}$$

For the potentials it is more useful to work in the Fourier space. The very very proud can try to continue working in the real space (ex14).

One writes the Lagrangian in the Fourier space

$$\begin{aligned} L &= \sum_i \frac{1}{2}m_i\dot{\vec{r}}_i^{\,2} + L_{em} \\ L_{em} &= \frac{\epsilon_0}{2}\int\left[|\vec{\mathcal{E}}(\vec{k})|^2 + c^2|\vec{\mathcal{B}}(\vec{k})|^2\right]d\vec{k} + \int\left[\vec{\mathcal{J}}^*(\vec{k}).\vec{\mathcal{A}}(\vec{k}) - \bar{\rho}^*(\vec{k})\bar{U}(\vec{k})\right]d\vec{k} \\ &= \int L_{k,em}d\vec{k} \end{aligned} \tag{108}$$

with

$$\begin{aligned} \vec{\mathcal{E}}(\vec{k}) &= -\frac{\partial\vec{A}(\vec{k})}{\partial t} - i\vec{k}\bar{U}(\vec{k}) \\ \vec{\mathcal{B}}(\vec{k}) &= i\vec{k}\times\vec{A}(\vec{k})\,. \end{aligned} \tag{109}$$

[6]We use the vectorial identity: $\vec{\nabla}(\vec{A}.\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} + (\vec{A}.\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}\times\vec{B})$.

The Lagrangian density is strictly local in $\vec{k}$: there are no derivatives of $\vec{\mathcal{A}}$ and $\bar{U}$ with respect to $\vec{k}$ it means that there is no coupling between neighbor points as in the real space.

Let us now write the Lagrange equations for the complex fields. For $\bar{U}(\vec{k})$ and $\bar{U}^*(\vec{k})$ we obtain (ex15) the Poisson's equation in the Fourier space $i\vec{k}.\vec{\mathcal{E}}(\vec{k},t) = \frac{\bar{\rho}(\vec{k},t)}{\epsilon_0}$. For $\vec{\mathcal{A}}_j(\vec{k})$ et $\vec{\mathcal{A}}_j^*(\vec{k})$ we obtain (ex16) the Ampère's equation (27). From (109) we obtain the Gauss (25) and Faraday (26) equations.

1.2.3 Classical Lagrangian in the *Coulomb gauge*

The Lagrangian L_{em} defined in (103) depends on 8 variables: $3A_j$, $3\dot{A}_j$, U and $\dot{U}$. However the field equations depend only on 6 variables: $3E_j$ et $3B_j$; Moreover, we have seen that $\vec{\mathcal{B}}_\| = 0$ and $\vec{\mathcal{E}}_\| = -\frac{i}{\epsilon_0}\frac{\bar{\rho}}{k}\hat{k}$. Therefore it only remains 4 transverse variables for which the equations of motion read

$$\frac{\partial \vec{\mathcal{B}}_\perp(\vec{k},t)}{\partial t} = -i\vec{k}\times\vec{\mathcal{E}}_\perp(\vec{k},t) \tag{110}$$

$$\frac{\partial \vec{\mathcal{E}}_\perp(\vec{k},t)}{\partial t} = ic^2\vec{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t) - \frac{1}{\epsilon_0}\vec{\mathcal{J}}_\perp(\vec{k},t)\ . \tag{111}$$

In order to reduce the Lagrangian (103) to 4 variables we eliminate the scalar potential and use the Coulomb gauge.

Elimination of $\bar{U}$

From the Poisson's equation $i\vec{k}.\vec{\mathcal{E}}(\vec{k},t) = \frac{\bar{\rho}(\vec{k},t)}{\epsilon_0}$ and $\vec{\mathcal{E}}(\vec{k},t) = -i\vec{k}\bar{U} - \frac{\partial\vec{\mathcal{A}}}{\partial t}$ we have $\bar{U} = \frac{1}{k^2}\left[ik\frac{\partial\mathcal{A}_\|}{\partial t} + \frac{\bar{\rho}}{\epsilon_0}\right]$. By plugging this expression in (108) one can show (ex17) that the Lagrangian is split into a transverse and a longitudinal part.

$$L = L_P + L_{em,\|} + L_{em,\perp} \tag{112}$$

with

$$L_P = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} \tag{113}$$

$$\begin{aligned} L_{em,\perp} &= -\int d\vec{k}\frac{\bar{\rho}^*\bar{\rho}}{\epsilon_0 k^2} + \epsilon_0\int d\vec{k}\left[\dot{\vec{\mathcal{A}}}_\perp^*\dot{\vec{\mathcal{A}}}_\perp - c^2k^2\vec{\mathcal{A}}_\perp^*\vec{\mathcal{A}}_\perp\right] \\ &+ \int d\vec{k}\left[\vec{\mathcal{J}}^*\vec{\mathcal{A}}_\perp + \vec{\mathcal{J}}\vec{\mathcal{A}}_\perp^*\right] \end{aligned} \tag{114}$$

and

$$\begin{aligned} L_{em,\|} &= \int\left[\mathcal{J}_\|^*\mathcal{A}_\| + \mathcal{J}_\|\mathcal{A}_\|^* - \frac{i}{k}(\bar{\rho}^*\dot{\mathcal{A}}_\| - \bar{\rho}\dot{\mathcal{A}}_\|^*)\right]d\vec{k} \\ &= \frac{d}{dt}\left\{\int\frac{i}{k}(\bar{\rho}\mathcal{A}_\|^* - \bar{\rho}^*\mathcal{A}_\|)d\vec{k}\right\}\ . \end{aligned} \tag{115}$$

As two Lagrangians that differ only by the total time derivative of a function of time are equivalent (it means that they lead to the same equations of motion) $L_{em,\|}$ is superfluous. Moreover, the Lagrange equation associated to $\mathcal{A}_\|$ gives $\dot{\rho} = -ik\mathcal{J}_\|$ which is a consequence of the charge conservation. In conclusion $\mathcal{A}_\|$ is not a real dynamical variable since it does not change the equations of motion of the system. We choose to work in the Coulomb gauge where ($\mathcal{A}_\| = 0$).

Lagrangian in the Coulomb gauge

$\vec{\nabla}.\vec{A} = 0 \Longrightarrow i\vec{k}.\vec{\mathcal{A}}(\vec{k}) = 0 \Longrightarrow \vec{\mathcal{A}}_\| = \vec{0}$. $\vec{A}$ is now purely transverse and $L_{em,\|} = 0$.

We finally have

$$L = \sum_i \frac{1}{2} m_i \vec{v}_i^2 - \int \frac{|\rho|^2}{\epsilon_0 k^2} d\vec{k} + \int L_{em,\perp}(\vec{k}) d\vec{k} \tag{116}$$

and

$$L_{em,\perp}(\vec{k}) = \epsilon_0 \left[\dot{\vec{\mathcal{A}}}^* \dot{\vec{\mathcal{A}}} - c^2 k^2 \vec{\mathcal{A}}^* \vec{\mathcal{A}} \right] + \left[\vec{\mathcal{J}}^* \vec{\mathcal{A}} + \vec{\mathcal{J}} \vec{\mathcal{A}}^* \right] \tag{117}$$

which also reads in the real space

$$L = \sum_i \frac{1}{2} m_i \vec{v}_i^2 - V_{Coulomb} + \int L_{em,\perp}(\vec{r}) d\vec{r} \tag{118}$$

with

$$L_{em,\perp}(\vec{r}) = \frac{\epsilon_0}{2} \left[\dot{\vec{A}}^2 - c^2 (\vec{\nabla} \times \vec{A})^2 \right] + \vec{j}.\vec{A} \tag{119}$$

and $\vec{\nabla}.\vec{A} = 0$.

1.2.4 Hamiltonian of the classical EM fields

Hamilton equations

The conjugate moments (linear momenta) associated to the variables x_j are $p_j \equiv \frac{\partial L}{\partial \dot{x}_j}$. The Lagrange equations give $\frac{dp_j}{dt} = \frac{\partial L}{\partial x_j}$ which leads to the definitions of the Hamiltonian and the Hamilton equations of motion

$$H(x_j, p_j) \equiv \sum_j \frac{dx_j}{dt} p_j - L \tag{120}$$

$$\dot{x}_j = \frac{\partial H}{\partial p_j} \tag{121}$$

$$\dot{p}_j = -\frac{\partial H}{\partial x_j} \, . \tag{122}$$

Conjugate moments in classical em

With $\vec{j} = \sum_i q_i \dot{\vec{r}}_i \delta(\vec{r} - \vec{r}_i)$ and (103) one obtains

$$(\vec{p}_i)_j = \frac{\partial L}{\partial (\dot{\vec{r}}_i)_j} = m_i (\dot{\vec{r}}_i)_j + q_i A_j(\vec{r}_i) \;, \tag{123}$$

the index j demotes the vectors components. Also, the associated moment (conjugate) to $\mathcal{A}_j(\vec{k})$ is

$$\pi_j(\vec{k}) \equiv \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}_j(\vec{k})} \right)^* = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}_j^*(\vec{k})} \;, \tag{124}$$

which leads to

$$\pi_j(\vec{k}) = \epsilon_0 \dot{\mathcal{A}}_j(\vec{k}) \tag{125}$$

or

$$\vec{\pi}(\vec{k}) = \epsilon_0 \dot{\vec{\mathcal{A}}}(\vec{k}) \tag{126}$$

and by using a Fourier transform

$$\Pi_j(\vec{r}) = \epsilon_0 \dot{A}_j(\vec{r}) \;. \tag{127}$$

or

$$\vec{\Pi}(\vec{r}) = \epsilon_0 \dot{\vec{A}}(\vec{r}) \;. \tag{128}$$

The hamiltonian reads

$$H = \sum_i \frac{1}{2m_i} \left[\vec{p}_i - q_i \vec{A}(\vec{r}_i) \right]^2 + V_{Coulomb} + \frac{\epsilon_0}{2} \int \left[\frac{\vec{\pi}^* \vec{\pi}}{\epsilon_0^2} + c^2 k^2 \vec{\mathcal{A}}^* \vec{\mathcal{A}} \right] d\vec{k} \tag{129}$$

and in the real space

$$H = \sum_i \frac{1}{2m_i} \left[\vec{p}_i - q_i \vec{A}(\vec{r}_i) \right]^2 + V_{Coulomb} + \frac{\epsilon_0}{2} \int \left[\frac{\vec{\Pi}^2}{\epsilon_0^2} + c^2 (\vec{\nabla} \times \vec{A})^2 \right] d\vec{r} \tag{130}$$

We arrive now at a very compact form of the equations of motion. We are now authorized to quantize.

Comment: This approach must be changed in the relativistic case since L which is defined in (103) is not invariant under the lorentz transformation [7]

Expressions of the fields as functions of the conjugate moments

$$\dot{\vec{r}}_i = \frac{1}{m_i}[\vec{p}_i - q_i \vec{A}(\vec{r}_i)] \tag{131}$$

$$\vec{E}(\vec{r}) = \vec{E}_{\|}(\vec{r}) + \vec{E}_{\perp}(\vec{r}) \tag{132}$$

[7] In this case one defines the action $I = \int \mathcal{L}(x) d^4x$ where x is the space-time quadrivector. By using ($c = 1$ in the following) as a definition $x^\mu = (t, \vec{r})$; $j^\mu = (\rho, \vec{j})$; $A^\mu = (U, \vec{A})$; $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = -F^{\mu\nu}$ and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ we have $\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu + \frac{1}{2}\left[(j_\mu A^\mu)^2 - \partial_\mu A^\nu \partial_\nu A^\mu \right]$ and the Maxwell equations read $\partial_\mu F^{\mu\nu} = j^\nu$ [equations (1) and (4)] and $\partial_\mu \tilde{F}^{\mu\nu} = 0$ [equations (2) and (3)].

$$\vec{E}_{\parallel}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} \tag{133}$$

$$\vec{E}_{\perp}(\vec{r}) = -\frac{1}{\epsilon_0}\Pi(\vec{r}) \tag{134}$$

$$\vec{B}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) \tag{135}$$

$$\vec{\mathcal{E}}(\vec{k}) = \vec{\mathcal{E}}_{\parallel}(\vec{k}) + \vec{\mathcal{E}}_{\perp}(\vec{k}) \tag{136}$$

$$\vec{\mathcal{E}}_{\parallel}(\vec{k}) = -\frac{i}{\epsilon_0}\frac{\bar{\rho}}{k}\hat{k} \tag{137}$$

$$\vec{\mathcal{E}}_{\perp}(\vec{k}) = -\frac{1}{\epsilon_0}\pi(\vec{k}) \tag{138}$$

$$\vec{\mathcal{B}}(\vec{k}) = i\vec{k}\times\vec{\mathcal{A}}(\vec{k}) \tag{139}$$

2 Quantum Electrodynamics

2.1 Quantization of the EM field from the harmonic oscillator

2.1.1 1D harmonic oscillator

Classical mechanics

We have

$$H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{140}$$

giving the equation of motion $m\frac{d^2x}{dt^2} = -\frac{dV}{dx}$. The solution is $x = x_0cos(\omega t - \varphi)$ with $\omega = \sqrt{\frac{k}{m}}$. The total energy is given by $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$. This quantity is conserved for an isolated system which implies that $E = \frac{1}{2}m\omega^2x_0^2$.

Quantum mechanics

We have

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 \tag{141}$$

$$\left[\hat{X}, \hat{P}\right] = i\hbar \tag{142}$$

$$\hat{H}|\varphi > = E|\varphi > \Leftrightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right]\varphi(x) = E\varphi(x) \ . \tag{143}$$

One defines the creation and annihilation operators as

$$\hat{\hat{X}} \equiv \sqrt{\frac{m\omega}{\hbar}}\hat{X} \ , \ \hat{\hat{P}} = \frac{1}{\sqrt{m\omega\hbar}}\hat{P} \ , \ \left[\hat{X}, \hat{P}\right] = i \tag{144}$$

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{\hat{X}} + i\hat{\hat{P}}) \ , \ \hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{\hat{X}} - i\hat{\hat{P}}) \tag{145}$$

$$\hat{\hat{X}} \equiv \frac{1}{\sqrt{2}}(\hat{a}^+ + \hat{a}) \ , \ \hat{\hat{P}} = \frac{1}{\sqrt{2}}(\hat{a}^+ - \hat{a}) \ . \tag{146}$$

One has the following properties

$$\left[\hat{a}, \hat{a}^+\right] = 1 \tag{147}$$

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) = \hbar\omega(\hat{a}\hat{a}^+ - 1/2) = \hbar\omega(\hat{N} + 1/2) \tag{148}$$

$$\hat{N} \equiv \hat{a}^+\hat{a} \ \textit{operator number of photons} \tag{149}$$

$$\left[\hat{N}, \hat{a}\right] = -\hat{a} \ , \ \left[\hat{N}, \hat{a}^+\right] = \hat{a}^+ \tag{150}$$

$$\hat{a}|\varphi_n > = \sqrt{n}|\varphi_{n-1} > \ , \ \hat{a}|\varphi_0 >= 0 \ , \ \hat{a}^+|\varphi_n >= \sqrt{n+1}|\varphi_{n+1} > \tag{151}$$

$$\hat{H}|\varphi_n > = E_n|\varphi_n > \ , \ E_n = (n + 1/2)\hbar\omega \tag{152}$$

$$|\varphi_n > = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|\varphi_0 > \ , \ \varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \tag{153}$$

$$\varphi_n(x) = \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{m\omega}{\hbar}x - \frac{d}{dx}\right]^n e^{-\frac{m\omega}{2\hbar}x^2} \ . \tag{154}$$

Standard deviations

$$\Delta X = \sqrt{<\hat{X}^2> - <\hat{X}>^2}\ ,\ \Delta P = \sqrt{<\hat{P}^2> - <\hat{P}>^2} \quad (155)$$

$$<\hat{X}> = 0\ ,\ <\hat{P}>=0\ ,\ \Delta X = \sqrt{(n+\frac{1}{2})\frac{\hbar}{m\omega}} \quad (156)$$

$$\Delta X \Delta P = (n+\frac{1}{2})\hbar \geq \frac{\hbar}{2}\ ,\ \Delta P = \sqrt{(n+\frac{1}{2})\hbar m\omega}\ . \quad (157)$$

Mean potential energy

$$<V(\hat{X})> = \frac{1}{2}m\omega^2 <\hat{X}^2>= \frac{E_n}{2}\ . \quad (158)$$

Mean kinetic energy

$$<\frac{\hat{P}^2}{2m}> = \frac{E_n}{2} =<V(\hat{X})> \quad \textit{Viriel's theorem.} \quad (159)$$

Contrary to the classical case the ground-state energy is not zero $E_0 = \frac{\hbar\omega}{2}$ (zero point energy). One cannot simultaneously have a zero kinetic energy and a zero potential energy.

2.2 Canonical quantization in the Coulomb gauge

2.2.1 Conjugate moments of the particles and the fields

We have seen that for the particles we have

$$(\vec{p}_n)_j = \frac{\partial L}{\partial(\dot{\vec{r}}_n)_j} = m_n(\dot{\vec{r}}_n)_j + q_n A_j(\vec{r}_n)\ , \quad (160)$$

then $\vec{p}_n = m_n\dot{\vec{r}}_n + q_n\vec{A}(\vec{r}_n)$ the conjugate moment associated to the particle n. For the fields, in the Coulomb gauge, the transverse fields are independent in the reciprocal space and we have $\vec{\mathcal{A}}_\perp(\vec{k}) = (\mathcal{A}_\epsilon(\vec{k}), \mathcal{A}_{\epsilon'}(\vec{k}))$. These two components are complex and we have $\vec{\mathcal{A}}_\perp(\vec{k}) \equiv \mathcal{A}_\epsilon(\vec{k})\hat{\epsilon} + \mathcal{A}_{\epsilon'}(\vec{k})\hat{\epsilon}'$ with $\hat{\epsilon}.\hat{k} = \hat{\epsilon}'.\hat{k} = \hat{\epsilon}.\hat{\epsilon}' = 0$. Let's remind that the unitary vectors $\hat{\epsilon}$ and $\hat{\epsilon}'$ can be *complex* in the case of a circular polarization. Let's also remind that within the Coulomb gauge $\vec{\mathcal{A}}_\parallel(\vec{k}) = 0$. Thus we have the following canonical commutation relations

$$\left\{ \begin{array}{l} \left[(\hat{\vec{r}}_n)_i, (\hat{\vec{p}}_m)_j\right] = i\hbar\delta_{ij}\delta_{nm} \\ \left[\hat{\mathcal{A}}_\epsilon(\vec{k}), \hat{\pi}_{\epsilon'}(\vec{k}')\right] = 0 \\ \left[\hat{\mathcal{A}}_\epsilon(\vec{k}), \hat{\pi}^+_{\epsilon'}(\vec{k}')\right] = i\hbar\delta_{\epsilon\epsilon'}\delta(\vec{k}-\vec{k}')\ . \end{array} \right. \quad (161)$$

We had introduced the normal variables $\vec{\alpha}(\vec{k},t)$ in order to establish an analogy between the EM field and harmonic oscillators. Recall:

$$\vec{\alpha}(\vec{k},t) \equiv -i\sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\vec{\mathcal{E}}_\perp(\vec{k},t) - c\hat{k}\times\vec{\mathcal{B}}_\perp(\vec{k},t)\right] \quad (162)$$

or

$$\vec{\mathcal{A}}_\perp(\vec{k},t) = \sqrt{\frac{\hbar}{2\epsilon_0\omega}}\left[\vec{\alpha}(\vec{k},t) + \vec{\alpha}^*(-\vec{k},t)\right] \ . \tag{163}$$

By using the reality of $\vec{E}(\vec{r},t)$ we have $\vec{\mathcal{E}}^*(\vec{k},t) = \vec{\mathcal{E}}(-\vec{k},t)$ which leads to

$$\vec{\alpha}(\vec{k},t) = \sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\omega\vec{\mathcal{A}}_\perp(\vec{k},t) - i\vec{\mathcal{E}}_\perp(\vec{k},t)\right] \ . \tag{164}$$

As $\vec{\mathcal{E}}_\perp(\vec{k}) = -\frac{1}{\epsilon_0}\pi(\vec{k})$ one has

$$\vec{\alpha}(\vec{k},t) = \sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\omega\vec{\mathcal{A}}_\perp(\vec{k},t) + \frac{i}{\epsilon_0}\vec{\pi}(\vec{k},t)\right] \ . \tag{165}$$

In the Coulomb gauge the symbol $\perp$ can be omitted. One defines

$$\hat{a}_\epsilon(\vec{k}) = \sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\omega\hat{\mathcal{A}}_\epsilon(\vec{k}) + \frac{i}{\epsilon_0}\hat{\pi}_\epsilon(\vec{k})\right] \tag{166}$$

$$\hat{a}^+_\epsilon(\vec{k}) = \sqrt{\frac{\epsilon_0}{2\hbar\omega}}\left[\omega\hat{\mathcal{A}}^+_\epsilon(\vec{k}) - \frac{i}{\epsilon_0}\hat{\pi}^+_\epsilon(\vec{k})\right] \ . \tag{167}$$

$\hat{a}_\epsilon(\vec{k}), \hat{a}^+_\epsilon(\vec{k})$ are the operators associated to the modes $\vec{k}, \epsilon$ of the harmonic oscillator and which verify the canonical commutation relations

$$\left[\hat{a}_\epsilon(\vec{k}), \hat{a}_{\epsilon'}(\vec{k}')\right] = 0 \tag{168}$$

$$\left[\hat{a}^+_\epsilon(\vec{k}), \hat{a}^+_{\epsilon'}(\vec{k}')\right] = 0 \tag{169}$$

$$\left[\hat{a}_\epsilon(\vec{k}), \hat{a}^+_{\epsilon'}(\vec{k}')\right] = \delta_{\epsilon\epsilon'}\delta(\vec{k}-\vec{k}') \ . \tag{170}$$

From now on the quantization of the field is immediate.

2.2.2 Quantum expressions of the fields and other quantities

From the expressions obtained previously we get [modes j of a cavity of volume V; the index j indicates $(\vec{k}_j, \hat{\epsilon}_j)$]

$$\hat{\vec{E}}_\perp(\vec{r}) = i\sum_j \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}\left[\hat{a}_j\hat{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} - \hat{a}^+_j\hat{\epsilon}^*_j e^{-i\vec{k}_j\cdot\vec{r}}\right] \tag{171}$$

$$\hat{\vec{B}}(\vec{r}) = i\sum_j \frac{1}{c}\sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}\left[\hat{a}_j\hat{k}_j\times\hat{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} - \hat{a}^+_j\hat{k}_j\times\hat{\epsilon}^*_j e^{-i\vec{k}_j\cdot\vec{r}}\right] \tag{172}$$

$$\hat{\vec{A}}_\perp(\vec{r}) = \sum_j \sqrt{\frac{\hbar}{2\omega_j\epsilon_0 V}}\left[\hat{a}_j\hat{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} + \hat{a}^+_j\hat{\epsilon}^*_j e^{-i\vec{k}_j\cdot\vec{r}}\right] \tag{173}$$

$$\hat{\vec{E}}_\parallel(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_n q_n\frac{\vec{r}-\hat{\vec{r}}_n}{|\vec{r}-\hat{\vec{r}}_n|^3} \tag{174}$$

with the commutation relations

$$\left[(\hat{\vec{r}}_n)_i, (\hat{\vec{r}}_m)_j\right] = \left[(\hat{\vec{p}}_n)_i, (\hat{\vec{p}}_m)_j\right] = 0 \tag{175}$$

$$\left[(\hat{\vec{r}}_n)_i, (\hat{\vec{p}}_m)_j\right] = i\hbar\delta_{ij}\delta_{nm} \tag{176}$$

$$\left[\hat{a}_i, \hat{a}_j\right] = \left[\hat{a}_i^+, \hat{a}_j^+\right] = 0 \tag{177}$$

$$\left[\hat{a}_i, \hat{a}_j^+\right] = \delta_{ij} \ . \tag{178}$$

In the two expressions at the beginning $i, j = x, y, z$ and, in the two expressions at the end, the indices i and j indicate the modes associated to the transverse fields $(\vec{k}_j, \hat{\epsilon}_j)$.

Transverse Hamiltonian and linear momentum

One has $(H_\perp = \frac{\epsilon_0}{2}\int(\vec{E}_\perp^2 + c^2\vec{B}^2)\ d\vec{r})$

$$\hat{H}_\perp = \sum_j \frac{\hbar\omega_j}{2}\left[\hat{a}_j^+\hat{a}_j + \hat{a}_j\hat{a}_j^+\right] = \sum_j \hbar\omega_j\left[\hat{a}_j^+\hat{a}_j + \frac{1}{2}\right] \tag{179}$$

and $(\vec{P}_\perp = \epsilon_0\int(\vec{E}_\perp \times \vec{B})\ d\vec{r})$

$$\hat{\vec{P}}_\perp = \sum_j \frac{\hbar\vec{k}_j}{2}\left[\hat{a}_j^+\hat{a}_j + \hat{a}_j\hat{a}_j^+\right] = \sum_j \hbar k_j \hat{a}_j^+\hat{a}_j \tag{180}$$

since $\sum_j \hbar\vec{k}_j/2 = \vec{0}$.

Total Hamiltonian

The total Hamiltonian reads

$$\hat{H} = \sum_n \frac{1}{2m_n}\left[\hat{\vec{p}}_n - q_n\hat{\vec{A}}(\hat{\vec{r}}_n)\right]^2 + \sum_n E^n_{Coulomb} + \frac{1}{8\pi\epsilon_0}\sum_{n\neq m}\frac{q_n q_m}{|\hat{\vec{r}}_n - \hat{\vec{r}}_m|}$$
$$+ \sum_j \hbar\omega_j\left[\hat{a}_j^+\hat{a}_j + \frac{1}{2}\right] \ . \tag{181}$$

Comment: The previous relations have been derived without considering the time dependence of the fields. It means that we are working in the framework of the Schrödinger's representation. In the Heisenberg's representation we have to take care of the fact that the commutation relations must be evaluated for the same time.

Space of states of the system "particles+fields"

We have $E = E_{part} \otimes E_{field}$. The operators $\hat{\vec{r}}_n$ and $\hat{\vec{p}}_n$ act only in E_{part}. $E_{field} = E_1 \otimes E_2 \otimes ... \otimes E_i \otimes ....$ E_i is the subspace associated to the harmonic oscillator i. A basis of E_i is, for example, the number of photons $\hat{N}_i = \hat{a}_i^+\hat{a}_i$. But it is not the only one...(coherent states, squeezed states...).

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